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Self-Similarity of the Desynchronization Process in a Network of Generalized Kuramoto Oscillators

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Abstract—The phenomenon of explosive synchronization in a network of generalized Kuramoto oscillators is considered. It has been shown that this process results from self-similarity observed after the loss of stability by synchronous clusters of different dimensions. The appearance of self-similarity can be revealed when studying the processes of destruction in the synchronous state of a network. It has been demonstrated that a system passes through a sequence of self-similar configurations of interacting oscillators after the abrupt destruction of a synchronous regime.

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The study of the dynamics of complex networks of phase oscillators provides the possibility to describe the behavior of a great number of real systems, such as neurons in the brain, myocardial cells, and electrical-power networks [1, 2]. The phenomenon of synchronization plays a key part in the collective dynamics of network unit elements, and the transitions between asynchronous and synchronous regimes are of great importance for the understanding of basic mechanisms in the behavior of interacting oscillators [3, 4] and complex networks [5–7].

From the thermodynamic viewpoint, a transition from asynchronous dynamics to a synchronous regime can be interpreted as a phase transition. There are two types of phase transitions: an abrupt transition to a synchronous state (the so-called "first-order transition") and a continuous phase transition ("second-order phase transition") [8]. Complex networks usually exhibit a smooth transition from asynchronous dynamics to synchronization with increasing parameter of coupling between network units [6, 9]. At the same time, a transition of another kind (first-order transition) called "explosive synchronization," when a network does not gradually pass through intermediate partially synchronized states, but abruptly transits from an asynchronous state into a completely synchronized regime by jump, is also observed in the dynamics of complex networks [10, 11]. It is known that explosive synchronization can appear in networks with different coupling topologies. A first-order transition to synchronous dynamics can occur in networks with a regular coupling topology, where each structural element of a network is coupled with all the other units [12], in networks with a random coupling topology [13], and in scale-free networks [14, 15], including scale-free networks characterized by time-delayed interelement couplings [16]. Explosive synchronization is also observed in networks of oscillators with adaptive couplings [17]. For a first-order transition (explosive synchronization) to appear, it is necessary to meet a number of conditions distinct for networks with different coupling topologies, and both the establishment and destruction of a synchronous regime in these cases are characterized by an abrupt qualitative change in the dynamics of network elements, and hysteresis may be observed in some cases [8, 14].

It is important to remark that, despite the fact that networks of Kuramoto oscillators [18, 19] are the most popular model for the study of explosive synchronization, a spontaneous instantaneously arising change in the dynamics of a network of oscillators with the resulting abrupt destruction/establishment of a synchronous regime may also be observed for other types of oscillators, when they are structural elements of complex networks, e.g., for piecewise-linear Rössler systems [10] or generalized Kuramoto oscillators [20]. In other words, the abrupt transition from a synchronous state of oscillators of a network with a complex topology of interelement couplings to asynchronous dynamics (and vice versa) represents a universal phenomenon appearing under certain conditions in complex networks with different unit elements and coupling topologies.

In this study, a network of generalized Kuramoto oscillators [20] is used as an example to demonstrate that an abrupt change in the state of a network with

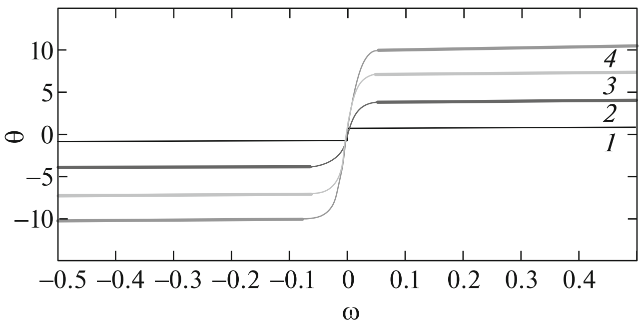

Fig. 1. Phases of generalized Kuramoto network oscillators (1) vs. their frequencies at $\lambda = 1.99 < \lambda_c$ and different time moments (*1*) $t_1 = 201$, (*2*) $t_2 = 318$, (*3*) $t_3 = 447$, and (*4*) $t_4 = 492$. Horizontal regions (solid line) correspond to the oscillators making up a synchronous cluster at a given time moment, and the fine lines correspond to the elements exhibiting an asynchronous dynamics.

a complex coupling topology results from the phenomenon of self-similarity, when the destruction of a synchronous regime occurs in a sequential order through self-similar configurations of interacting oscillators, which lose stability at the same control parameter characterizing the intensity of interelement couplings.

The results presented in this work were obtained for a network of generalized Kuramoto oscillators [20] with $N = 5 \times 10^3$ elements, the dynamics of which is described as

$$\dot{\theta}_i = \omega_i + \frac{\lambda|\omega_i|}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1,\dots,N, \quad (1)$$

where θ_i and ω_i are the instantaneous phase and eigenfrequency of the ith oscillator, the point means the time derivative, and λ is the coupling parameter. In the considered model, all the network oscillators are coupled with each other, the eigenfrequencies of oscillators ω_i were equidistantly specified in the interval [−0.5, 0.5], and the initial phases of oscillators were selected equal to zero without loss of generality. Since all the eigenfrequencies of modified Kuramoto oscillators are different, synchronous dynamics appears in the network only at a coupling parameter exceeding its boundary value λ_c, but all the network oscillators are in synchronism in this case, and the network itself as a whole may be considered as an integrated synchronous cluster with dimension N. The destruction of a synchronous cluster begins at critical coupling parameter λ_c (for the considered network, $\lambda_c = 2.0$) [20].

When a synchronous cluster with dimension N loses stability with an increase in the coupling parameter λ and λ^*, some oscillators come out of synchronism, while the oscillators remaining in a synchronous state compose a cluster with dimension $K < N$. In the case in which a new cluster proves to be stable at the same coupling parameter λ^*, the network gradually passes into a new stable state, whose destruction requires to decrease the intensity of couplings again, corresponding to a second-order phase transition [8]. Correspondingly, to observe a first-order phase transition, during which the synchronous state of the entire network is abruptly destructed with the transition to a non-synchronous dynamics of oscillators without large clusters, all the synchronous structures of different dimensions K such that $K < N$ must become unstable at the same coupling intensity λ_c. In other words, when the process of destruction in the synchronous state of a network is started at $\lambda = \lambda_c$, some oscillators begin to evolve in an asynchronous fashion and, as a consequence, the primary coherent structure with dimension N is replaced by a coherent structure (composed of $K(t)$ synchronous oscillators, $K(0) = N$) with a smaller dimension, which in turn is also unstable and replaced by a coherent structure $K(\tau) < K(t)$, $t < \tau$, etc. Since the synchronous clusters of all the dimensions $K \le N$ lose stability at the same coupling intensity, they must have self-similarity properties. Hence, a coherent cluster of synchronous oscillators sequentially passes through self-similar configurations with a decreasing dimension $K(t)$ when an "explosive" transition from synchronization to asynchronous dynamics takes place, and all these configurations become unstable at the same coupling parameter $\lambda = \lambda_c$.

The self-similar mechanism implemented during a first-order phase transition in a network of generalized Kuramoto oscillators can be revealed via the temporal analysis of the destruction process in a synchronous cluster at a coupling parameter, which is slightly lower than its critical value. The dependences of phases θ_i on frequencies ω_i for the elements of studied network (1) at different time moments $t_1 < t_2 < t_3 < t_4$ and $\lambda = 1.99$ are shown in Fig. 1. The horizontal regions correspond to the oscillators composing a synchronous cluster at each of the considered time moments, and the points in the inclined regions correspond to the oscillators that have come out of synchronism. As can be seen, the dimension of a synchronous cluster decreases with time and, correspondingly, the number of asynchronous oscillators grows, but the remaining synchronous oscillators form structures that are self-similar to each other.

Hence, a network of generalized Kuramoto oscillators passes through a sequence of self-similar configurations of interacting oscillators, which cause the abrupt destruction of a synchronous regime, during a first-order phase transition. It is important to note that destruction begins after a certain threshold and is not chaotic, but occurs through a sequence of decreasing ordered structures.

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