

**Self-similarity in explosive synchronization of complex networks**Alexey A. Koronovskii,<sup>\*</sup> Maria K. Kurovskaya,<sup>†</sup> and Olga I. Moskalenko<sup>‡</sup>  
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We report that explosive synchronization of networked oscillators (a process through which the transition to coherence occurs without intermediate stages but is rather characterized by a sudden and abrupt jump from the network's asynchronous to synchronous motion) is related to self-similarity of synchronous clusters of different size. Self-similarity is revealed by destructing the network synchronous state during the backward transition and observed with the decrease of the coupling strength between the nodes of the network. As illustrative examples, networks of Kuramoto oscillators with different topologies of links have been considered. For each one of such topologies, the destruction of the synchronous state goes step by step with self-similar configurations of interacting oscillators. At the critical point, the invariance of the phase distribution in the synchronized cluster with respect to the cluster size is reported.

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Networks of phase oscillators offer a benchmark description in a large variety of natural systems, such as neurons in the human brain, cardiac pacemaker cells, power grids, etc. [1]. On the other hand, synchronization of such networked oscillators is often the basis for the emergence of collective dynamics. Synchronization is a universal nonlinear phenomenon [2] and means the adjustment of rhythms of coupled oscillators due to their interaction. It may be observed in different forms, such as phase synchronization [3–5], generalized synchronization [6–8], time scale synchronization [9], lag synchronization [10–12], complete synchronization [13–16], etc. We here concentrate on the phase synchronization regime, where an entrainment of phases (and, accordingly, frequencies) of chaotic signals takes place. Therefore, taking into account the prominence of phase synchronization, the transition between asynchronous and synchronous states is of fundamental importance for understanding some core mechanisms through which interacting oscillators adjust their pace and phases [10,17,18], and collective dynamics occurs [19–23].

From a thermodynamics point of view the passage from asynchronous to synchronous oscillations (and vice versa) can be considered as a phase transition. Phase transitions to the synchronous state may be abrupt (as in the case of explosive synchronization, resembling a first-order transition) or continuous (like a second-order phase transition) [24]. Typically, smooth phase transitions take place when the coupling strength between nodes grows and the asynchronous

oscillatory motion becomes unstable [19,22,25]. However, explosive synchronization (ES, where the network does not pass through intermediate partial synchronization stages but rather jumps suddenly from the asynchronous to the synchronized motion and vice versa) can also be observed in complex networks [26–28], including the case of the presence of external fields [29].

ES has been observed for different network architectures, including networks with all-to-all coupled elements [30], random networks [31], networks with scale-free topology [27,32,33] (including scale-free networks with time-delayed coupling [34]), and networks of adaptively coupled oscillators [35]. Furthermore, ES has been studied for many different kinds of networked oscillators, from Kuramoto [36,37], to modified Kuramoto oscillators [38], to chaotic piecewise Rössler units [26]. Because of its ubiquity, ES can be considered a general feature of coupled oscillator networks. In ES, both the establishment (the forward transition) and the destruction (the backward transition) of the synchronous state are abrupt, and in several cases the overall scenario features hysteretic loops, which are the hallmark of irreversibility.

We stress that a smooth (second-order-like) transition is typical in complex networks, while abrupt jumps from the asynchronous to asynchronous motion (and vice versa) are rarer phenomena which may be observed under some specific circumstances. For ES to be realized, certain specific conditions (that are distinct for different network topologies) must be fulfilled. For regular cliques ES may take place [38] for specific realizations of a uniform distribution in the natural frequencies [30]. In turn, in Ref. [28] certain constraints on the frequency differences between each node and its neighbors are imposed explicitly to avoid the formation of a clustering process. As far as scale-free networks (namely, Barabási-Albert network) is concerned, a correlation between the

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natural frequency of each oscillator and its degree,  $\omega_i = F(k_i)$  may support ES [27,32]. Additionally, also adaptive coupling may enhance ES in networks [35]. Nevertheless, the several observations of ES phenomena call for their generalities, despite their rare manifestations.

A very relevant question is whether there are universal features accompanying ES, i.e., situation which may be common to different network topologies and/or dynamics. Stimulated by this challenge, we here study the destruction mechanisms of the synchronous state in networks with practically all typical link topologies where ES is known to be observed: random networks with evenly spaced natural frequencies (following Refs. [36,37]), modified Kuramoto models (see Refs. [38,43]), and scale-free networks described in Refs. [27,32]. Our results show that, in all considered cases, ES is connected with self-similarity in the stability loss of the synchronous clusters of different size.

The structure of our paper is the following. In Sec. II we consider (both analytically and numerically) the destruction process of the fully synchronized state in random network of Kuramoto oscillators, and we reveal the core self-similar properties of the abrupt transition from the synchronous to asynchronous dynamics. Section III is devoted to report on ES in the modified Kuramoto model, whereas the scale-free network case is considered in Sec. IV. The final summary and remarks are given in Sec. V.

## II. RANDOM NETWORKS

We start with a random network of Kuramoto oscillators [36,37] whose dynamics is given by

$$\dot{\varphi}_i = \omega_i + \frac{\lambda}{N} \sum_{j=1}^N a_{ij} \sin(\varphi_j - \varphi_i), \quad (1)$$

where  $N$  is the number of coupled oscillators,  $\varphi_i$  and  $\omega_i$  are the instantaneous phase and natural frequency of  $i$ th oscillator, respectively,  $\lambda$  is the coupling strength, and  $\{a_{ij}\}$  are the elements of the adjacency matrix that uniquely defines the nodes' interactions ( $a_{ij} = a_{ji} = 1$  if oscillators  $i$  and  $j$  are connected with each other and zero otherwise). The natural frequencies  $\omega_i$  are supposed to be different and, therefore, the synchronized motion appears only above some coupling strength threshold  $\lambda_c$ . We consider the case of evenly spaced natural frequencies,

$$\omega_i = -\Omega + \frac{\Omega}{N}(2i - 1), \quad (2)$$

where  $\Omega = 0.5$ ,  $i = 1, \dots, N$ , just in the same way as it was done in Refs. [36,37]. In other words, the frequency distribution  $g(\omega)$  should be considered as symmetric and centered at zero:

$$g(\omega) = \begin{cases} \frac{1}{2\Omega} & \text{for } |\omega| \leq \Omega \\ 0 & \text{for } |\omega| > \Omega. \end{cases} \quad (3)$$

The adjacency matrix of an Erdős-Rényi (ER) random graph [39] has been obtained by the well-known algorithm which consists in connecting each couple of nodes with a probability  $0 < p < 1$  [1]. For  $p = 1$ , network Eq. (1)

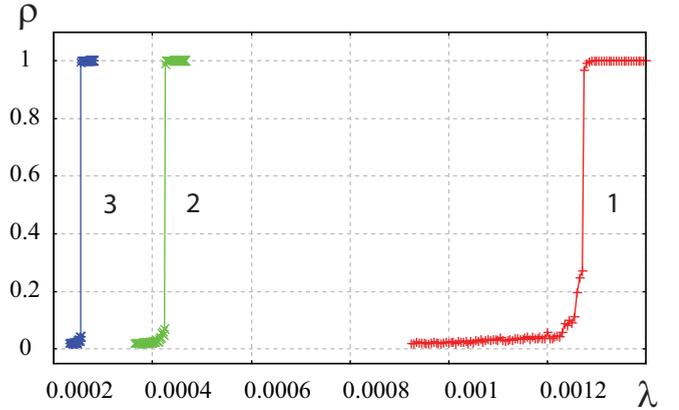


FIG. 1. The fraction of synchronous oscillators in the network,  $\varrho = \mathcal{N}/N$  (where  $\mathcal{N}$  is the synchronous cluster size), vs. the coupling strength  $\lambda$ . Results are obtained for Erdős-Rényi random networks of size  $N = 10^3$  (curve 1),  $N = 3 \times 10^3$  (curve 2), and  $N = 5 \times 10^3$  (curve 3),  $p = 0.5$ .

becomes a clique, which was studied in Ref. [30]. In our work, we have used  $N = 5 \times 10^2$ ,  $10^3$ , and  $5 \times 10^3$  elements, with probabilities  $p = 0.3, 0.5, 0.9$ , and all numerical calculations have confirmed our findings.

By examining the backward transition, it is seen that the synchronous state loses its stability abruptly at  $\lambda_c^N$  (see Fig. 1). Above  $\lambda_c^N$ , all network oscillators are synchronized, and the entire network can be considered as a unique, giant cluster of size  $N$  [40]. At the coupling threshold (or, more precisely, just below it, i.e., for  $\lambda \rightarrow \lambda_c^N -$ ), the synchronous cluster starts being destroyed, with a sudden collapse of the synchronous state taking place. The symbols  $N$  and  $\mathcal{N}$  are used to denote the size of the network and of synchronous cluster, respectively, as well as we denote by  $k_i$  ( $k_i^N$ ) the total number of links, or node degree (the links to oscillators in the synchronous cluster) of the  $i$ th network's element.

Under the assumption of a large number of network oscillators,  $N \rightarrow \infty$  (in fact, in the thermodynamic limit of an infinite population), the evolution law of the oscillators Eq. (1) at critical point  $\lambda_c^N$  within the synchronous cluster of size  $\mathcal{N}$  may be rewritten in the form

$$\dot{\varphi}_i = \omega_i + \lambda_c^N \kappa_i \varrho \int_{-\pi/2}^{\pi/2} R_{\mathcal{N}}(\psi) \sin(\psi - \varphi_i) d\psi, \quad (4)$$

where  $R_{\mathcal{N}}(\varphi)$  is the probability distribution of phases of synchronized oscillators,  $0 \leq \kappa_i \leq 1$  is the normalized number of links that the  $i$ th node forms with synchronized oscillators,  $\kappa_i = k_i^N / \mathcal{N}$ , and  $\varrho = \mathcal{N}/N$  is the portion of synchronized oscillators in the network. At the critical point  $\lambda_c$ , one can consider two groups of oscillators: the synchronous and the asynchronous (drifting) ones. If one further suppose that the destruction of the synchronous cluster develops in time more slowly than the drift of the asynchronous oscillators, only the contribution of the synchronized oscillators should be accounted for in Eq. (4).

Now, for large  $N$  (assuming also  $Np \gg 1$ ), the degree distribution may be approximated by a Dirac function as  $G(k) = \delta(k - \langle k \rangle)$ , where  $\langle k \rangle = p$ . The quantity  $\langle k \rangle$  may be

substituted for  $\kappa$  in Eq. (4), and the density of synchronized nodes with phase  $\varphi$ ,  $R_{\mathcal{N}}(\varphi)$  may be found as

$$R_{\mathcal{N}}(\varphi) = g_s(\omega) \frac{d\omega}{d\varphi}, \quad (5)$$

where  $g_s(\omega)$  is the frequency distribution of the synchronous oscillators. As for synchronized oscillators  $\dot{\varphi} = 0$ , Eq. (4) gives

$$\omega + \lambda_c^{\mathcal{N}} \langle k \rangle \varrho \int_{-\pi/2}^{\pi/2} R_{\mathcal{N}}(\psi) \sin(\psi - \varphi) d\psi = 0, \quad (6)$$

or

$$\frac{d\omega}{d\varphi} = \lambda_c^{\mathcal{N}} \langle k \rangle \varrho \int_{-\pi/2}^{\pi/2} R_{\mathcal{N}}(\psi) \cos(\psi - \varphi) d\psi = 0. \quad (7)$$

As a consequence, the relation for  $R_{\mathcal{N}}(\varphi)$  is

$$R_{\mathcal{N}}(\varphi) = g_s(\omega) \lambda_c^{\mathcal{N}} \langle k \rangle \varrho \int_{-\pi/2}^{\pi/2} R_{\mathcal{N}}(\psi) \cos(\psi - \varphi) d\psi. \quad (8)$$

At  $\lambda_c$ , a part of the oscillators starts moving asynchronously, and the coherent structure of size  $N$  is replaced by a smaller cluster [consisting of  $\mathcal{N}(t)$  synchronous oscillators,  $\mathcal{N}(0) = N$ ]. Note that the integral Eq. (8) remains valid for any number of synchronous oscillators  $\mathcal{N}$ . At the time at which the destruction of the synchronous cluster starts, the integral Eq. (8) may be written for the whole network (i.e.,  $\mathcal{N} = N$ ,  $g_s(\omega) = g(\omega)$ ,  $\kappa = k/N$ ,  $\varrho = 1$ ,  $\lambda_c^{\mathcal{N}} = \lambda_c^N$ ),

$$R(\varphi) = \Lambda \int_{-\pi/2}^{\pi/2} R(\psi) \cos(\psi - \varphi) d\psi, \quad (9)$$

where  $\Lambda = \Lambda^N = \lambda_c^N \langle k \rangle / (2N\Omega)$ ,  $R(\varphi) = R_{\mathcal{N}}(\varphi)$ . Equation (9) is a homogeneous Fredholm integral equation of the second kind and admits solution only for

$$\Lambda = \Lambda_c = \frac{2}{\pi} \quad (10)$$

(see Ref. [41]), which in our case takes the form

$$R(\varphi) = \frac{1}{2} \cos \varphi. \quad (11)$$

In other words, Eqs. (10) and (11) describe the state of network at  $\lambda_c$ , i.e., at the beginning of the abrupt transition from synchronized to asynchronous dynamics.

According to Eq. (10), the synchronous state loses its stability at

$$\lambda_c^N = \frac{4N\Omega}{\pi \langle k \rangle} = \frac{4\Omega}{p\pi}, \quad (12)$$

since  $\langle k \rangle = Np$ , and the density of the synchronized oscillators with the phase  $\varphi$  is given by Eq. (11). For the chosen parameter values ( $\Omega = 0.5$ ,  $p = 0.5$ ,  $N = 5 \times 10^3$ ), one has  $\lambda_c^N \approx 1.273$ . In the limit of  $p \rightarrow 1$ , Eq. (12) coincides with the critical value obtained for the all-to-all connected network [30], as well as with the value where the incoherent solution becomes unstable according to the classical result [42] for all unimodal distributions,  $\lambda_c = 2/\pi g(0)$ .

When the fully synchronized state starts to be destroyed, several oscillators begin drifting. One can suppose that the synchronous cluster loses first the oscillators whose frequencies are closer to the boundaries  $\pm\Omega$ . In other words, the fully synchronized state starts being replaced by a synchronous

cluster consisting of  $\mathcal{N}$  oscillators whose frequencies are bounded by  $\pm\Gamma$  ( $\Gamma = \varrho\Omega$ ) and distributed as

$$g_s(\omega) = \begin{cases} \frac{1}{2\Gamma} & \text{for } |\omega| \leq \Gamma \\ 0 & \text{for } |\omega| > \Gamma. \end{cases} \quad (13)$$

The criticality properties of the newly formed cluster of size  $\mathcal{N}$  are also described by Eq. (9) with  $\Lambda = \Lambda^{\mathcal{N}} = \lambda_c^{\mathcal{N}} \langle k \rangle \varrho / (2\Gamma)$ ,  $R(\varphi) = R_{\mathcal{N}}(\varphi)$ . Remarkably, the quantity  $\Lambda^{\mathcal{N}}$  does not depend on  $\mathcal{N}$ , and it is invariant for all configuration of the synchronous oscillators, i.e.,  $\Lambda^{\mathcal{N}} = \Lambda^N$ . As a consequence, the arisen synchronous cluster of size  $\mathcal{N}$  also becomes unstable at  $\lambda_c^{\mathcal{N}} = \lambda_c^N = \lambda_c$ , with the probability distribution of synchronized network oscillators,  $R_{\mathcal{N}}(\varphi)$ , being governed by the same regularity Eq. (11).

In other words, when the abrupt transition from the synchronization to asynchronous dynamics takes place, the coherent cluster of synchronous oscillators passes sequentially through different self-similar configurations of size  $\mathcal{N}(t)$ , with all of them becoming unstable at once, at the same critical point  $\lambda_c$ . The probability distributions  $R_{\mathcal{N}}(\varphi)$  of the instantaneous phases  $\varphi$  also demonstrate the same self-similarity properties. The profile of the phase distributions for synchronized nodes remains unchanged at the fixed value of the coupling strength  $\lambda \rightarrow \lambda_c$ , i.e.,

$$R_{\mathcal{N}}(\varphi) = R_N(\varphi) = R(\varphi), \quad \forall \mathcal{N} \leq N, \quad (14)$$

where  $R(\varphi)$  is governed by Eq. (11), for the considered case of random networks and given frequency distribution Eq. (3). Importantly, Eq. (14) reflects a self-similarity property and, in fact, is the marker of the abrupt transition from synchronous to asynchronous network state. If a smooth, second-order-like transition occurs (as, e.g., for the case when the distribution of frequencies  $g(\omega)$  is not compact), Eq. (14) does not hold, and a self-similar behavior of synchronous clusters is not observed.

To illustrate the obtained results, we have simulated numerically the process of the destruction of the synchronous state. The control parameters of the model have been selected as  $N = 5 \times 10^3$ ,  $\Omega = 0.5$ ,  $p = 0.5$ ,  $\lambda = \lambda_c = 1.273$ . The abrupt transition from coherence to incoherence is shown in Fig. 2, where the evolution of the distributions of the synchronized oscillators over frequencies [Fig. 2(a)] and phases [Fig. 2(b)] are reported. The size of the synchronous cluster  $\mathcal{N}(t)$  decreases linearly with time [except for the small final stage of cluster destruction ( $t > 700$ ); see Fig. 2(c)]. One can see that the synchronous oscillators are distributed evenly and uniformly over frequencies  $\omega \in [-\Gamma(t), \Gamma(t)]$  until the very end of the process of the synchronous cluster destruction [see Fig. 2(a)], whereas the profile of the phase distribution,  $\mathcal{N}_{\varphi}(\varphi, t)$ , remains practically unchanged with only its amplitude decreasing [see Fig. 2(b)].

To prove the invariance of the probability distribution of the synchronized oscillators Eq. (14) we have calculated  $R_{\mathcal{N}(t)}(\varphi)$  at different moments of time. The results are reported in Fig. 3. One can see that probability density for synchronized oscillators is indeed invariant, and well agrees with the analytical curve Eq. (11) confirming the theoretical prediction Eq. (14).

Summarizing the results described so far, we have to stress two core features:

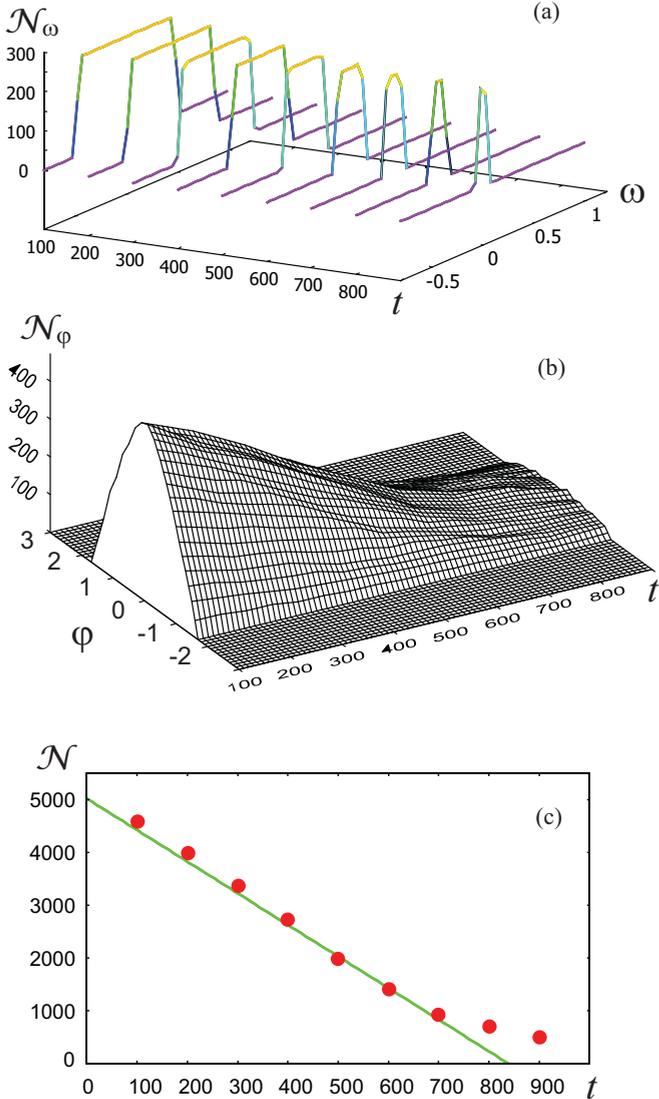


FIG. 2. (a)  $\mathcal{N}_\omega(\omega, t)$ , (b)  $\mathcal{N}_\phi(\phi, t)$ , and (c)  $\mathcal{N}(t)$  during the process of destruction of the synchronous state.  $N = 5 \times 10^3$ ,  $p = 0.5$ ,  $\lambda_c = 1.273$ .

(A) The backward transition is associated with a well-defined self-similar behavior: when the transition from synchronization to incoherence takes place, the coherent cluster of synchronous oscillators passes sequentially through different self-similar configurations of size  $\mathcal{N}(t)$ , with all of them starting being destroyed at once and at the same critical point  $\lambda_c$ . It is this feature that makes possible a first-order-like phase transition in the network. If indeed only one of the newly formed synchronous clusters of size  $\mathcal{N}$  would have been stable, an abrupt transition would have been impossible and, instead, a smooth second-order-like transition would have been realized.

(B) All configurations of the synchronized oscillators that arise and disappear through the abrupt transition are characterized by a self-similar (invariant) probability density that may be formalized with the help of self-similarity condition Eq. (14).

These two features are likely to appear also for different network architectures where ES is possible. Statement (A)

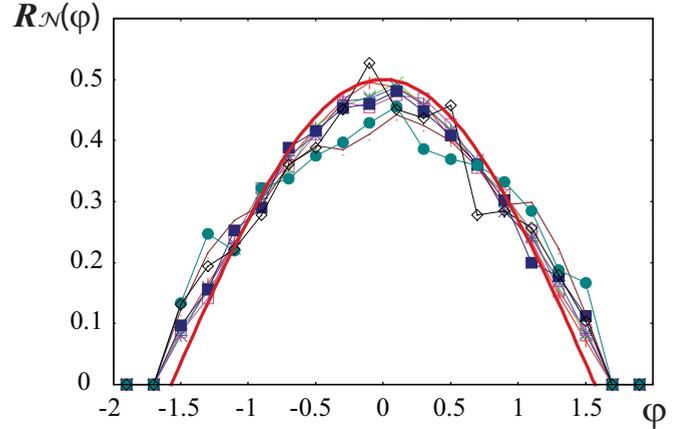


FIG. 3.  $R_N(\phi)$  (see text for definition), calculated at eight different moments in time (from the very beginning till the very end of the backward transition).  $N = 5 \times 10^3$ ,  $p = 0.5$ ,  $\lambda_c = 1.273$ . The theoretically predicted distribution,  $R(\phi)$ , given by Eq. (11) is shown by a bold solid (red) curve, the numerically obtained data are shown by points:  $t = 100$ —symbols “+”,  $t = 200$ —“x”,  $t = 300$ —“\*”,  $t = 400$ —“□”,  $t = 500$ —“■”,  $t = 600$ —“-”,  $t = 700$ —“•”,  $t = 800$ —“◇”.

explains clearly the mechanisms underlying the abrupt character of the transition, statement (B) gives a tool to check the presence of self-similarity in cases when an analytical examination is not possible.

### III. MODIFIED KURAMOTO MODEL

The modified Kuramoto model proposed in Ref. [43] and studied in detail in Ref. [38] completely confirms our findings. The evolution of the network is now governed by

$$\dot{\phi}_i = \omega_i + \frac{\lambda|\omega_i|}{N} \sum_{j=1}^N \sin(\phi_j - \phi_i), \quad i = 1, \dots, N, \quad (15)$$

where the frequency distribution has been taken to be uniform and even, according to Ref. [38]. This model displays ES with hysteresis, at variance with the random network considered in the previous Section, with the forward and backward transitions being distinct. We here concentrate again on the backward transition.

The modified Kuramoto model allows for an analytical treatment of both the forward and backward transitions. The backward phase transition is known to take place at  $\lambda_c = 2$ , which depends on neither the size of network nor the type of the frequency distribution [38].

Without loss of generality, the phase distribution of the oscillators for  $\lambda > 2$  may be described by

$$R(\phi) = \frac{1}{2}[\delta(\phi - \Theta) + \delta(\phi + \Theta)], \quad (16)$$

where  $\delta(\cdot)$  is the Dirac  $\delta$  function. In other words, the synchronous oscillators are split evenly into two symmetric clusters located at  $\phi_{\pm} = \pm\Theta$ . With the decrease of the coupling strength  $\lambda$ , these two clusters move gradually towards to critical phases  $\phi_{c\pm} = \pm\pi/4$  and at the critical point  $\lambda_c$  the

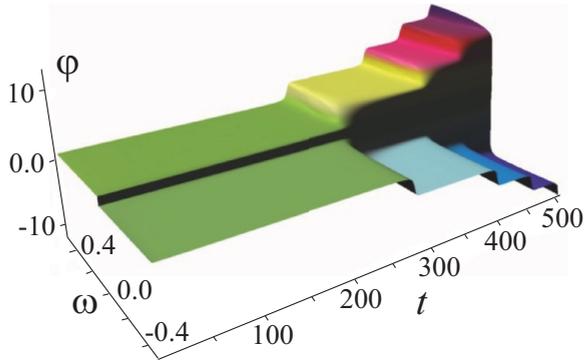


FIG. 4. The instantaneous phases,  $\varphi$ , vs.  $\omega$  and time, during the process of destroying the synchronous state in the generalized Kuramoto model Eq. (15).  $N = 5 \times 10^3$ ,  $\lambda_c = 2$ .

synchronous state of network starts being destroyed, with  $\Theta = \pi/4$  (see Ref. [38] for details).

The abrupt transition from synchronous to asynchronous state is shown in Fig. 4. One can see clearly the sequence of self-similar synchronous clusters that appear sequentially, whose size  $\mathcal{N}$  decreases in time. The phases of the oscillators arranged in the synchronous clusters are  $\varphi = \pm\pi/4 \pm 2\pi n$ , ( $n \in \mathbb{N}$ ) (except for the very short transient episodes when one synchronous structure is replaced by another one). Once again, both features (A) and (B) are revealed during the backward transition.

#### IV. SCALE-FREE NETWORK

The next point is revealing the presence of features (A) and (B) for ES in a scale-free topology. In this latter case, following Refs. [27] and [32], we consider a modified version of the Kuramoto network, where the natural frequency of each node is taken to be equal to its node degree, i.e.,  $\omega_i = k_i$ . As a consequence, the frequency distribution coincides with the degree distribution. A scale-free topology of links is considered, implemented with the help of the approach proposed by Barbási and Albert [44] with  $N = 10^3$  nodes and  $\langle k \rangle = 6$ . The degree distribution is governed by a power law,  $P(k) \sim k^\gamma$ , with  $\gamma = -3$ . The frequency distribution is therefore  $g(\omega) \sim \omega^\gamma$ .

The backward transition takes place at  $\lambda_c \approx 1.28$ . The distribution of the phases  $\varphi$  as a function of the degree  $k$  for a Barbási-Albert network is given in Refs. [32]. As a consequence, the relationship between frequency and phase of each Kuramoto oscillator may be rewritten in the form

$$\omega = \frac{\langle k \rangle}{1 - \lambda r \sin \varphi}, \quad (17)$$

where  $r$  is the order parameter [45,46]. Therefore, using Eq. (5) and taking into account that at the critical point one has  $\lambda_c r = 1$ , the critical probability density may be written as

$$R(\varphi) = \frac{1}{2}(1 - \sin \varphi) \cos \varphi. \quad (18)$$

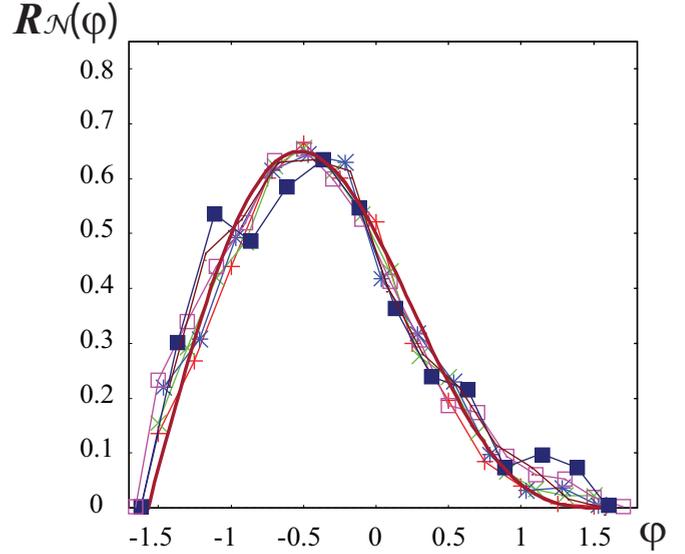


FIG. 5. The probability densities of the synchronized oscillators,  $R_{\mathcal{N}}(\varphi)$ , calculated for the different moments of time (from the very beginning till the very end of the destruction process) for the abrupt collapse of the synchronous state of scale-free network of Kuramoto oscillators,  $N = 10^3$ ,  $p = 0.5$ . The theoretically predicted probability distribution,  $R(\varphi)$ , governed by Eq. (18) is shown by the bold solid (red) curve, the numerically obtained data are shown by points.

The obtained density Eq. (18) obeys the normalization condition

$$\int_{-\pi/2}^{\pi/2} R(\varphi) d\varphi = 1. \quad (19)$$

Remarkably, it does not depend on the mean degree of the network,  $\langle k \rangle$ , i.e.,  $R(\varphi)$  is invariant for all scale-free networks with  $\gamma = -3$ . Therefore, to seek for the presence of criterium (B), one has to compare  $R_{\mathcal{N}}(\varphi)$ , calculated at different moments of time (from the very beginning till the very end of the destruction process) and estimated at  $\lambda_c$ , together with the theoretical prediction Eq. (18), just in the same way as it was done in Fig. 3.

The comparison is shown in Fig. 5, where an excellent agreement between theoretical and numerical data is obtained. The numerically calculated distributions fit the analytical curve, and start deviating only at the end of the destruction process when the number of synchronous elements becomes too small. In other words, the probability densities for all configurations that arise and disappear during the backward transition are invariant, and the features (A) and (B) are also observed in this case.

#### V. CONCLUSIONS

In summary, we have shown that explosive synchronization in complex networks of oscillators is connected with self-similarity of the synchronous clusters of different size. More precisely, we have shown that the destruction of the graph's synchronous state goes step by step with self-similar configurations of synchronous clusters of interacting oscillators,

with the self-similarity condition Eq. (14), which takes place at the critical point of coupling strength. All newly formed synchronous clusters of smaller size start being destroyed at once and at the same critical point for an abrupt transition to be realized. If this requirement is not satisfied and even one arisen synchronous cluster would be stable, a smooth transition would be realized. From the mathematical point of view it means that the self-similarity condition Eq. (14) is not valid for the case of a smooth transition.

The invariance of the phase distribution in the synchronized cluster with respect to the cluster size at the critical point has been revealed both theoretically and numerically. Our results are specified with Kuramoto oscillators for practically all

known network examples where the explosive synchronization takes place, but we expect that the same mechanisms should be observed in networks of other oscillator models as well as other topologies of links and frequency distributions. These findings provide a fresh and novel insight into the mechanisms of explosive synchronization, and are of value for both the theory of complex networks and the practical applications of ES in a wide spectrum of human activities.

#### ACKNOWLEDGMENT

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