Numerical and analytical investigation of the chimera state excitation conditions in the Kuramoto-Sakaguchi oscillator network

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ABSTRACT

In this paper we study the conditions of chimera states excitation in ensemble of non-locally coupled Kuramoto-Sakaguchi (KS) oscillators. In the framework of current research we analyze the dynamics of the homogeneous network containing identical oscillators. We show the chimera state formation process is sensitive to the parameters of coupling kernel and to the KS network initial state. To perform the analysis we have used the Ott-Antonsen (OA) ansatz to consider the behavior of infinitely large KS network.

Keywords: Chimera state, complex network, Kuramoto-Sakaguchi network

1. INTRODUCTION

In present, the study on dynamical systems collective behavior focused a great extent to the analysis of chimera states.[1–5] This phenomena arises in ensembles of identical oscillator models that are abundant in physics and biology. Mentioned chimera states, characterized by simultaneous existence of spatial regions of coherent and incoherent subgroups in a network, have been first discovered in 2002 by Kuramoto and Battogtokh.[1] They have found chimera states in a network of non-locally coupled nonlinear media described by the complex Ginzburg-Landau system[6] and then demonstrated in a network of Kuaramoto-Sakaguchi phase model.[7] However, in recent studies, it has been shown that such a chimera-like states can emerge in networks of oscillators with global coupling,[8] nearest neighbor local coupling[9] too. Recent studies further confirm that chimera states are not limited to phase oscillators only, it can emerge in limit cycle and chaotic systems,[10] neural systems,[11, 12] time-discrete map,[13] and Boolean networks.[14] As a result, amplitude-mediated chimera has been proven[3] to be a reality which encourages experimental verification and in fact, it has now been experimentally evidenced in chemical,[15] electronic,[16] electrochemical[17] and opto-electronic[18] systems.

However, studies of chimera states were confined so far to the domain of single-layer network. Among many other effects associated with the emergence of chimera states, important and less studied topics are the stability of chimera states,[19] how different isolated networks individually be in any of the states, chimera, coherent and even incoherent, may be affected when they interact with each other,[20] how intra-coupling parameters influence the process of spatial coherent patterns formation? Its consideration along with a theoretical investigation demands due attention for prospective practical use.[21] In this paper we address this issue of chimera states in a framework of dependence of chimera state formation under the variation of different controlling parameters, namely parameters of coupling between the nonlinear elements of the network. We also study how different types of initial conditions effect on the chimera-like pattern formation. The main interest of this work is devoted to the methods of controlling the processes of chimera state formation and related phenomena.

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2. MODEL

We carry out an analytical study of the non-local network of phase oscillators to show the conditions of chimera patterns formation. So, in the framework of the provided analysis the processes do not depend on the number of oscillators in the KS network. We restrict our analysis to the network of non-locally coupled oscillators with rectangular coupling kernel. We have applied the Ott-Antonsen (OA) approach[22] to reduce infinite-dimensional network of Kuaramoto-Sakaguchi phase model to the systems of two-dimensional differential equations. Although, OA anzatz is generally used in the case of coupled non-identical oscillators with different natural frequencies, this method is also effectively applied to the analysis of homogenous networks.[23–26]

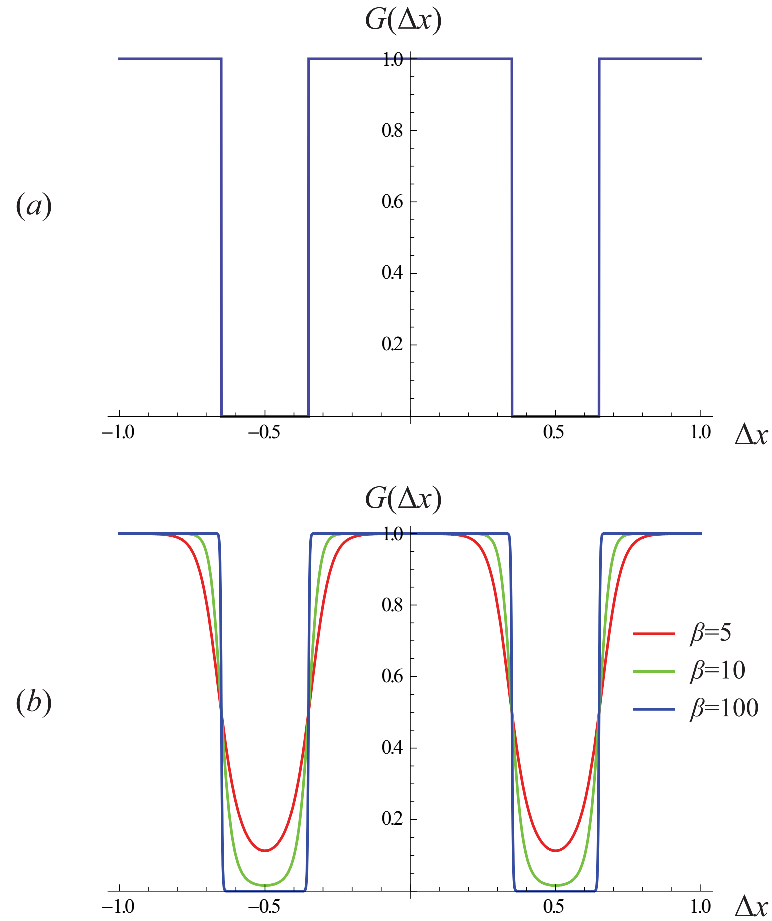

Figure 1. Plots of the coupling kernel form $G(\Delta x)$ defined by equation (5) – (a) and by equation (6) – (b).

In our theoretical work we have used KS equations in the following form:

$$\frac{d\varphi_i}{dt} = \omega_i - \frac{\lambda}{2R+1} \sum_{k=i-R}^{i+R} \sin(\varphi_i - \varphi_k + \alpha), \tag{1}$$

where φ_i, ω_i – phase and eigenfrequency of KS oscillator, R, λ – coupling radius and coupling strength, α is a controlling parameter. In this work we have fixed $\alpha = 1.45$ and $\lambda = 0.085$.

In the framework of this approach we describe the dynamics of the network in terms of probability density function (PDF) $f(x, \varphi, t)$. As the number of oscillators is constant in time we can write the continuity equation for PDF,

$$\frac{\partial f}{\partial t}+\frac{\partial\left(fv\right)}{\partial\varphi}=0, \tag{2}$$

where v is,

$$v=\frac{d\varphi}{dt}=\omega-\frac{1}{2\imath}\{re^{\imath\varphi}-\bar{r}e^{-\imath\varphi}\}, \tag{3}$$

the overbar indicates complex conjugate and r is the order parameter written in the following form,

$$r=e^{\imath\alpha}\frac{\lambda}{2R}\int_0^1 G\left(x-y\right)\int_0^{2\pi} f\left(y,\varphi,t\right)e^{-\imath\varphi}d\varphi dy, \tag{4}$$

where $G(x-y)$ is the rectangular coupling kernel as used in,

$$G\left(x-y\right)=H\left(\cos\left(\left(x-y\right)2\pi\right)-\cos\left(R^{1,2}2\pi\right)\right). \tag{5}$$

The form of this coupling kernel is plotted in Fig. 1(a). The coupling kernel $G\left(x-y\right)$ written in this form perfectly fits the way of intra-layer non-local coupling in our numerical simulation. $H\left(x\right)$ in (5) is the Heaviside step function. In our analytical treatment we replace this kernel with differentiable function:

$$G\left(x-y\right)=\frac{1}{1+e^{-\beta(\cos((x-y)2\pi)-\cos(R2\pi))}}. \tag{6}$$

The form of this kernel depends on the value of parameter β (Fig. 1)(b). In the limit of $\beta\to\infty$ function (6) tends to (5). In this work we have fixed parameter $\beta=100$.

We are looking for the solution $f\left(x,\varphi,t\right)$ in the form of Fourier series taking into account the OA ansatz $f_n\left(x,\varphi,t\right)=a\left(x,\varphi,t\right)^n$:

$$f\left(x,\varphi,t\right)= \\ =\frac{1}{2\pi}\left(1+\sum_{n=1}^{\infty}a^n\left(x,\varphi,t\right)e^{\imath n\varphi}+c.c.\right) \tag{7}$$

Substituting (3)-(7) to (2) we obtain the final equation,

$$\frac{\partial a}{\partial t}+\imath\omega a+\frac{1}{2}[\bar{r}a^2-r]=0 \tag{8}$$

where the order parameter with respect to OA ansatz becomes

$$r\left(x,t\right)=\frac{\lambda}{2R}\int_0^1 G\left(x-y\right)a\left(y,\varphi,t\right)dy. \tag{9}$$

To determine the phase distribution we substitute OA ansatz $a=|a|e^{-j\psi}$ to (7):

$$f\left(x,\varphi,t\right)= \\ =\frac{1}{2\pi}\frac{1-\{|a|\}^2}{(1-|a|)^2+2|a|\left(1-\cos[\varphi-\psi]\right)}. \tag{10}$$

Here $|a|$ is a maximum value of the phase distribution and ψ is a phase value corresponding to the distribution maximum.

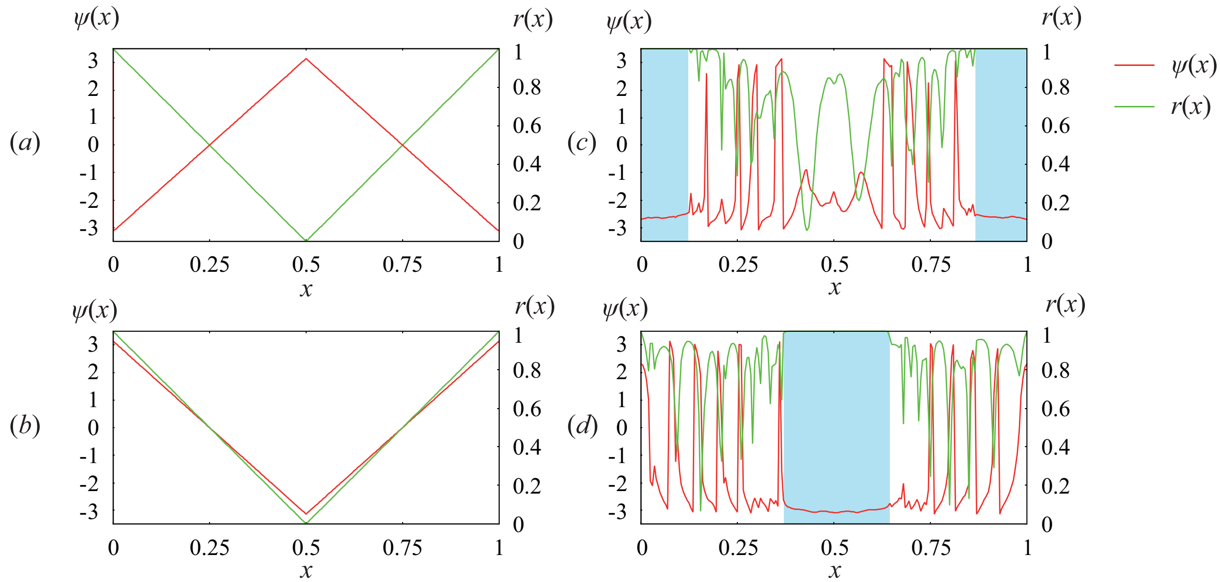

Figure 2. Distributions of order parameter $r(x)$ and phase $\psi(x)$. Here, plots (a), (b) present the initial state of the KS network and plots (c), (d) correspond to the network state after transient process starting from (a) and (b) respectively. Coherent subpopulation is highlighted with blue.

3. RESULTS

Let us discuss the results of the theoretical treatment of KS oscillators network with non-local coupling. In Fig. 2 the distributions of order parameter $r(x)$ and phase $\psi(x)$ are presented. Here, left column corresponds to initial distribution and right column corresponds to final state of the system after transient process.

Initial conditions in Fig. 2(a) are chosen in the following form:

$$\psi(x) = \begin{cases} \pi(4x-1), & x \in \left[0, \frac{1}{2}\right], \\ \pi(3-4x), & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{11}$$

And in Fig. 2(b):

$$\psi(x) = \begin{cases} \pi(1-4x), & x \in \left[0, \frac{1}{2}\right], \\ \pi(4x-3), & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{12}$$

Initial distribution of order parameter $r(x)$ has the same form for both cases:

$$r(x) = \begin{cases} 1-\dfrac{x}{2}, & x \in \left[0, \frac{1}{2}\right], \\ 2x-1, & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{13}$$

One can observe, the emergence of chimera state in both cases and the location of coherent subpopulation of the network corresponds to the minimum of $\psi(x)$ distribution.

Also, we have analyzed the case of randomly distributed initial phase $\psi(x)$. The results are shown in Fig. 3. It is easy to see, that the chimera state is suppressed when the initial phase $\psi(x)$ is distributed in the wide range, namely from $-\pi$ to π (Fig. 3(a),(b)). When the range of initial ψ values becomes more narrow from -0.5π to 0.5π (Fig. 3(c),(d)) one can observe the excitation of dislocated chimera state with respect to one plotted in Fig. 2(b). Starting from $[-0.3\pi, 0.3\pi]$ distribution we obtain the typical chimera state with the coherent pattern in the middle of x-space (Fig. 3(e),(f)). Notable, that even in the case of small range of randomly distributed phase $\psi(x)$ the KS network evolves to the stable chimera state (Fig. 3(g),(h)).

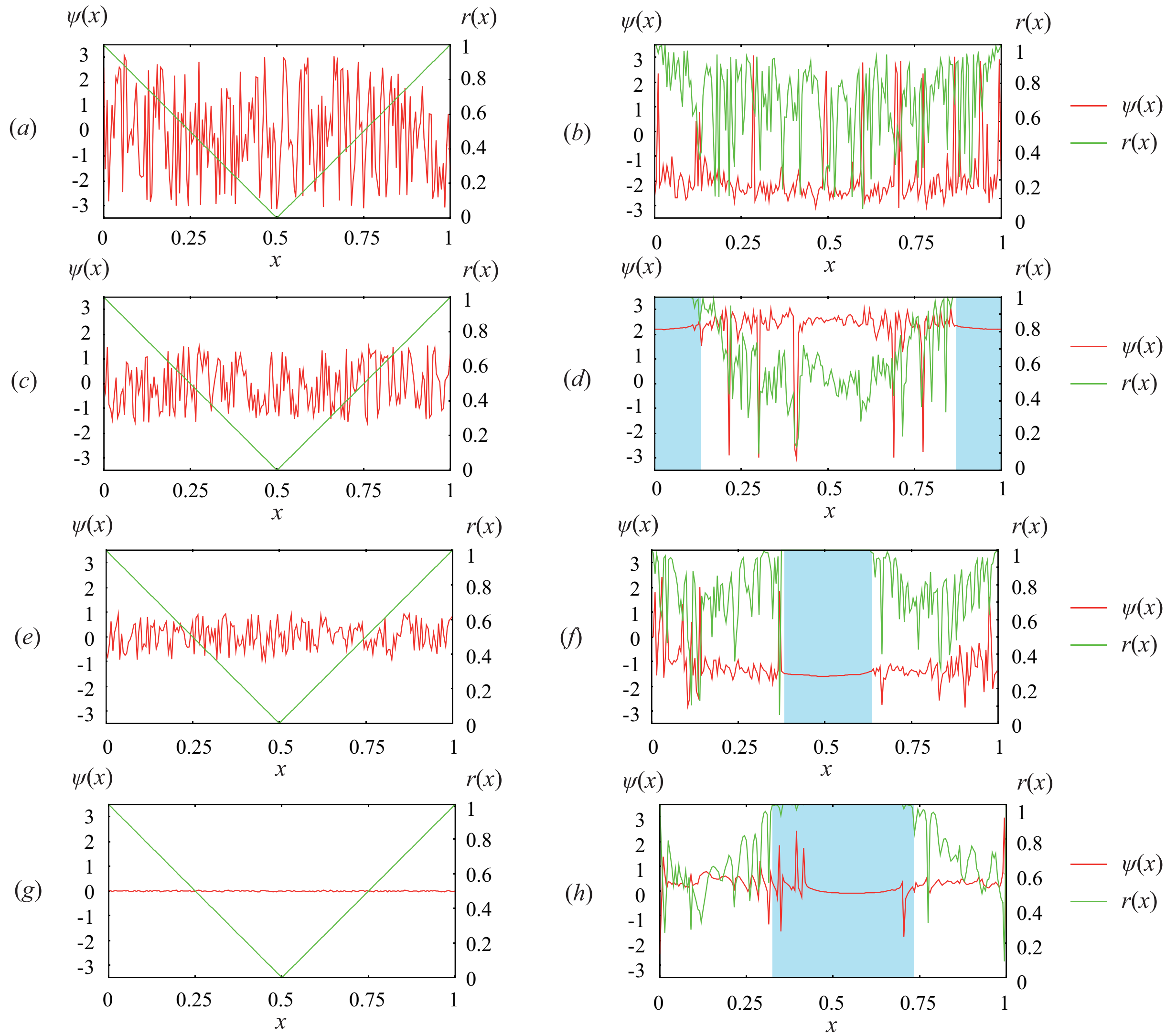

Figure 3. The initial (a),(c),(e),(g) and final (b),(d),(f,(h) distributions of KS network in the case of randomly distributed phase $\psi(x)$. Here, in (a),(b) – ψ is randomly distributed in $[-\pi, \pi]$; in (c),(d) – ψ is randomly distributed in $[-0.5\pi, 0.5\pi]$; in (e),(f) – ψ is randomly distributed in $[-0.3\pi, 0.3\pi]$; in (g),(h) – ψ is randomly distributed in $[-0.01\pi, 0.01\pi]$. Coherent subpopulation is highlighted with blue.

Finally, we have observe the KS network behavior under the variation of coupling radius R. We can vary coupling radius R from 0 to 0.5, where 0 corresponds to the absence of coupling (empty coupling area) and 0.5 corresponds to global coupling (all-to-all coupling). At the small values of coupling radius R one can see the suppression of chimera state that corresponds to totally incoherent behavior (Fig. 4(a)). The increase of R gives birth to the chimera state regime, that is characterised by two narrow coherent areas (Fig. 4(b)). Further increase of R transforms this chimera state behavior to the typical one with the only coherent subpopulation (Fig. 4(c)). At the values close to 0.5 we obtain the regime of almost coherent behavior in the whole KS network (Fig. 4(d)).

4. CONCLUSIONS

We have explored chimera states formation conditions in a framework KS network analysis by means of Ott-Antonsen approach. We have shown that the chimera state formation process is sensitive to the parameters of

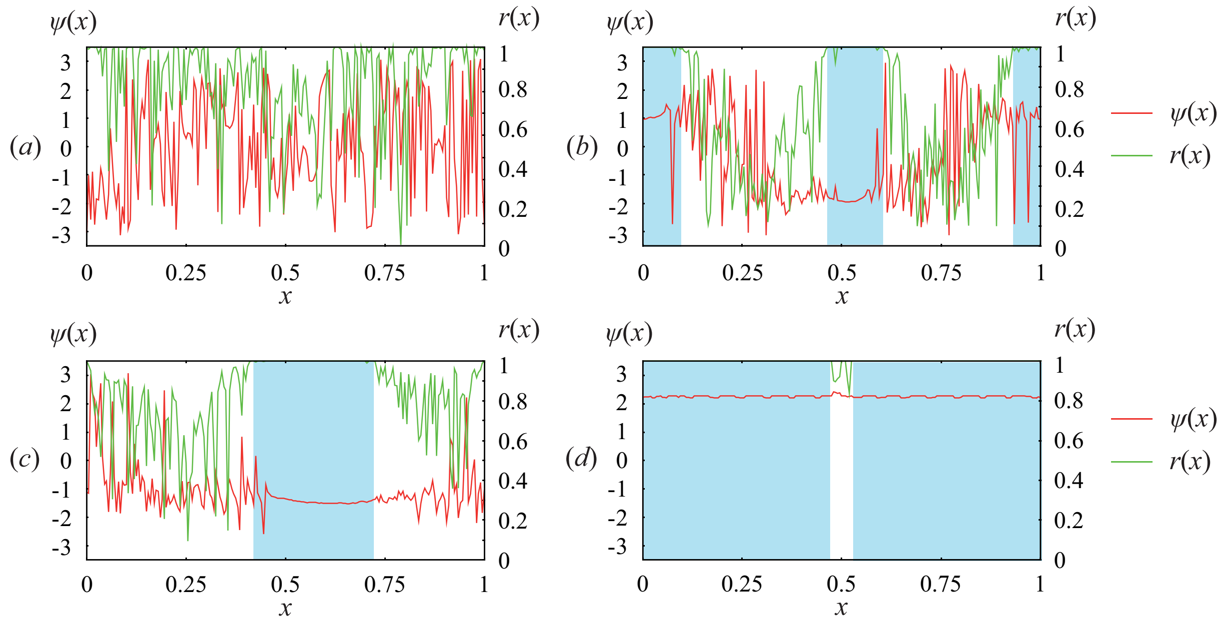

Figure 4. The final distributions of KS network under the variation of coupling radius R. (a) – $R = 0.05$,(b) – $R = 0.2$, (c) – $R = 0.35$, (d) – $R = 0.48$. Coherent subpopulations are highlighted with blue.

coupling kernel and to the KS network initial state. In fact, we have detected the following regularities: the width of coherent subpopulation in the regime of chimera state formation depends on the peak value of initial phase distribution in case of linear phase distribution and also depends on the width of the phase distribution in case of randomly distributed initial phase. The important point is the dependence of chimera state formation regime on the radius of coupling area – the more is the value of coupling radius, the wider is the coherent subpopulation.

5. ACKNOWLEDGMENTS

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REFERENCES

[1] Kuramoto, Y., "Self-entrainment of a population of coupled nonlinear oscillators," *Lect. Notes in Physics* **30**, 420 (1975).

[2] Strogatz, S.H., "From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D.* **143**, 1 (2000).

[3] Sethia, G.C., Sen, A. and Johnston, G.L., "Amplitude-mediated chimera states" *Phys. Rev. E* **88**, 042917 (2013).

[4] Omelchenko, I., Zakharova, A., Hövel, P., Siebert, J. and Schöll, E., "Nonlinearity of local dynamics promotes multi-chimeras" *Chaos* **25**, 083104 (2015).

[5] Mishra, A., Hens, C., Bose, M., Roy, P.K. and Dana S.K., "Chimeralike states in a network of oscillators under attractive and repulsive global coupling" *Phys. Rev. E* **92**, 062920 (2015).

[6] Kuramoto, Y. and Nakao, H., "Origin of Power-Law Spatial Correlations in Distributed Oscillators and Maps with Nonlocal Coupling" *Phys. Rev. Lett.* **76**, 4352 (1996).

[7] Sakaguchi, H. and Kuramoto, Y., "A soluble active rotater model showing phase transitions via mutual entertainment" *Prog. Theor. Phys.* **76**, 576 (1986)

[8] Schmidt, L. and Krischer, K., "Clustering as a Prerequisite for Chimera States in Globally Coupled Systems" *Phys. Rev. Lett.* **114**, 034101 (2015).

[9] Bera B.K., Ghosh D. and Lakshmanan, M., "Chimera states in bursting neurons" *Phys. Rev. E* **93**, 012205 (2016).

[10] Gu, C., St-Yves, G. and Davidsen, J., "Spiral Wave Chimeras in Complex Oscillatory and Chaotic Systems" *Phys. Rev. Lett* **111**, 134101 (2013).

[11] Hizanidis, J., Kanas, V., Bezerianos, A. and Bountis, T., "Chimera states in networks of nonlocally coupled HindmarshRose neuron models" *Int. J. Bifurcat. Chaos* **24**, 1450030 (2014).

[12] Bera, B.K., Ghosh, D. and Banerjee, T., "Imperfect traveling chimera states induced by local synaptic gradient coupling" *Phys. Rev. E* **94**, 012215 (2016).

[13] Omelchenko, I., Maistrenko, Y. L., Hovel, P. and Scholl, E., "Loss of Coherence in Dynamical Networks: Spatial Chaos and Chimera States" *Phys. Rev. Lett* **106**, 234102 (2011).

[14] Rosin, D.P., Rontani, D., Haynes, N.D. and Scholl, E., Gauthier, D.J., "Transient scaling and resurgence of chimera states in networks of Boolean phase oscillators" *Phys. Rev. E* **90**, 030902(R) (2014).

[15] Tinsley, M.R., Nkomo, S. and Showalter, S. "Chimera and phase-cluster states in populations of coupled chemical oscillators" *Nat. Phys.* **8**, 662 (2012).

[16] Larger, L., Penkovsky, B. and Maistrenko, Y.L., "Virtual Chimera States for Delayed-Feedback Systems" *Phys. Rev. Lett.* **111**, 054103 (2013).

[17] Wickramasinghe, M. and Kiss, I.Z., "Spatially organized dynamical states in chemical oscillator networks: synchronization, dynamical differentiation, and chimera patterns" *PLoS ONE* **8**, e80586 (2013).

[18] Hagerstrom, A., Murphy, T.E. and Roy, R., Hövel, P., Omelchenko, I., Schöll, E., "Experimental observation of chimeras in coupled-map lattices" *Nat. Phys.* **8**, 658 (2012).

[19] Sieber J., Omelchenko O.E. and Wolfrum M., "Controlling Unstable Chaos: Stabilizing Chimera States by Feedback" *Phys. Rev. Lett.* **112**, 054102 (2014).

[20] Maksimenko, V. A., Makarov, V. V., Bera, B. K., Ghosh, D., Dana, S. K., Goremyko, M. V., Frolov, N. S., Koronovskii, A. A. and Hramov, A. E., "Excitation and suppression of chimera states by multiplexing" *Phys. Rev. E* **94**, 052205 (2016).

[21] Panaggio, M.J. and Abrams, D.M., "Chimera states: coexistence of coherence and incoherence in networks of coupled oscillators" *Nonlinearity* **28**, R67 (2015).

[22] Ott E. and Antonsen T.M., "Low dimensional behavior of large systems of globally coupled oscillators" *Chaos* **18**, 037113 (2008).

[23] Marvel, S.A., Mirollo, R.E. and Strogatz S.H., "Identical phase oscillators with global sinusoidal coupling evolve by mobius group action" *Chaos* **19**(4), 043104 (2009).

[24] Xu, C., Xiang, H., Gao, J. and Zheng, Z., "Collective dynamics of identical phase oscillators with high-order coupling" *Scientific Reports* **6**, 31133 (2016).

[25] Bordyugov, G., Pikovsky, A. and Rosenblum, M., "Self-emerging and turbulent chimeras in oscillator chains" *Phys. Rev. E* **82**, 035205 (2010).

[26] Hong, H. and Strogatz, S. H., "Conformists and contrarians in a kuramoto model with identical natural frequencies" *Phys. Rev. E* **84**, 046202 (2011).