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THEORETICAL AND MATHEMATICAL PHYSICS

Analysis of the Stability of States of Semiconductor Superlattice in the Presence of Tilted Magnetic Field

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Abstract—A method to calculate the spectrum of the Lyapunov exponents for a periodic semiconductor nanostructure (superlattice) described in the framework of a semiclassical approach is proposed. The analysis of the stability of a stationary state in such a system is performed for autonomous dynamics and in the presence of a tilted magnetic field. The method of the Lyapunov exponents is used to study the effect of the tilted magnetic field on the stability of the stationary state and the characteristics of subterahertz oscillation regimes.

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INTRODUCTION

Semiconductor superlattices represent nanostructures consisting of thin (several nanometers) periodic layers of semiconductor materials with different band gaps and almost equal periods of the crystalline lattice [1–3]. A periodic modulation of the conduction band in such systems gives rise to energy minibands in which the electron transport is accompanied by various nonlinear effects [4] that are interesting for both fundamental physics and practical applications.

From the practical point of view, semiconductor superlattices are promising systems for the generation [5] and amplification [6] of terahertz signals. It is known that dc voltage applied to semiconductor superlattice causes the development of instability and the formation of domains of charges (a region with an increased concentration of electrons) that drift along the system and induce oscillations of current that flows in the system [7, 8].

With regard to the application of superlattices for the generation of sub-THz and THz radiation, it is of interest to study the stability of the stationary state of the semiconductor nanostructure. In addition, it is expedient to develop methods for analysis and classification of oscillation regimes in such a system.

Normally, modern analysis of the stability of stationary states of semiconductor nanostructures employs the NL criterion [9], which has been proposed in [10] for the Gunn diode. However, such an algorithm can be used in the approximation of the uniform distribution of the electric field in the structure. In several systems with nonuniform electric field, the algorithm leads to significant errors in the calculation of the applied voltage that is needed for the development of instability. In addition, the criterion cannot be used for the analysis of semiconductor superlattices in the presence of external effects (modulation of the applied voltage, magnetic field, and resonant systems). However, the analysis of such effects is important for the study of the dynamics of semiconductor structures in sub-THz and THz electronic devices. In particular, magnetic field can be used to efficiently control the properties of electron transport and the analysis with allowance for the magnetic field is important for an increase in the generation frequency in semiconductor structures [11]. It is also important to take into account the effect of external resonant systems on the generation of oscillations in semiconductor nanostructures [12].

To study the stability of and analyze the oscillation regimes in a semiconductor structure for autonomous dynamics and with allowance of various external effects, it is expedient to employ the Lyapunov exponents. Such a method is actively used in similar problems for a set of flow dynamic systems and discrete mappings [13]. However, the method for calculation of the spectrum of the Lyapunov exponents has been proposed and tested predominantly for systems with a relatively small number of the degrees of freedom. The direct application of such a method in the analysis of spatially distributed systems (e.g., semiconductor superlattices) is impossible. The main problem related to the calculation of the spectrum of Lyapunov exponents for spatially distributed systems is related to the infinite-dimensional phase space in which the states of such systems are determined, analysis of distributed perturbations, simulation of their dynamics, and orthogonalization. Attempts at the calculation of the spectrum of Lyapunov exponents based on artificial sampling [14] of distributed systems or simulation of perturbed state [15] are inefficient, since the features related to the spatial distribution are disregarded. Note that such features may cause atypical dynamics related to modifications of the original system.

In this work, we propose a method for the calculation of the spectrum of Lyapunov exponents for a semiconductor superlattice based on the analysis of the dynamics of a set of small perturbations for the ground state. Such an approach is used to analyze the stationary state of the system and the corresponding oscillation regimes for autonomous dynamics and under the action of a tilted magnetic field.

SYSTEM UNDER STUDY

To describe the cooperative transportation of charge in semiconductor superlattices, we use the semiclassical approach of [16]. In the framework of such an approach, the motion of charge carriers and the spatiotemporal dynamics of the configuration of the electric field in the structure are calculated with the aid of a hydrodynamic model containing self-consistent equations of continuity and Poisson equations:

$$\frac{\partial F}{\partial x} = \nu(n - n_D), \quad \frac{\partial n}{\partial t} = -\beta\frac{\partial J}{\partial x}, \tag{1}$$

where $F(x, t)$ is the electric field distribution, $n(x, t)$ is the concentration of carriers, $J(x, t)$ is the current density in the semiconductor structure, $\nu = 15.769$ and $\beta = 0.031$ are dimensionless control parameters, and $n_D = 1.0$ is the dimensionless equilibrium concentration of carriers in the semiconductor. Dimensionless quantities in Eqs. (1) are related to the dimensional parameters:

$$n = n'/n'_D, \quad x = x'/L', \quad \nu = L'en'_D/(F'_C\varepsilon'_0\varepsilon'_r),$$
$$F'_C = \hbar/(ed'\tau'), \quad t = t'/\tau', \quad J = J'/(en'_D\vartheta'_0), \tag{2}$$
$$\vartheta'_0 = \delta\Delta'd'/(2\hbar), \quad \beta = \vartheta'_0\tau'/L'.$$

Here, dimensional quantities are denoted with prime. For the system under study, $n'_D = 3 \times 10^{22}$ m^{-3} is the equilibrium concentration of carriers; $L' = 115.2$ nm and $d' = 8.3$ nm are the length and period of the superconductor superlattice, respectively; $e > 0$ is the electron charge; $\tau' = 250$ fs is the scattering time of carriers in the semiconductor; $\varepsilon'_r = 12.5$ is the relative permittivity of the material; and $\Delta' = 19.1$ meV is the width of the energy miniband.

The drift approximation is used to calculate the current density

$$J(x,t) = n(x,t) \times \upsilon(F(x,t)), \tag{3}$$

where $\upsilon(F)$ corresponds to the dimensionless drift velocity of carriers in the semiconductor superlattice [17]. In the framework of the semiclassical approach, such a dependence can be obtained using the law of motion of a single electron in the miniband of the semiconductor superlattice with allowance for the scattering time [18]. At low temperatures, in the absence of external magnetic fields, the dependence is represented as

$$\upsilon(F) = \frac{F}{1 + F^2}. \tag{4}$$

Dimensionless potential difference $V = V'/(F_cL)$ at the boundaries of the system serves as the control parameter in the model. Such a difference remains constant:

$$V = V_c + \int_0^1 F(x)dx, \tag{5}$$

where V_c is the voltage drop at the contacts of the semiconductor superlattice that is chosen in accordance with the parameters of the experimental sample [16, 18].[1]

Figure 1 presents the results of the numerical simulation of Eqs. (1) and (3)–(5) for two different values of the applied voltage. Figure 1a illustrates the spatiotemporal dynamics of the concentration of carriers for the dimensionless potential difference $V \sim 9$, which corresponds to a voltage of $V' = 330$ mV. It is seen that the transient process in the system results in the stationary distribution of the concentration that is characterized by a near-emitter layer with an increased concentration of carriers. Figure 1c shows the time dependence of the current density. Figures 1b and 1d present the spatiotemporal dynamics and the dependence of the current density for $V \sim 11$ ($V' = 400$ mV). In this case, the system exhibits the development of instability and the formation of moving domains (Fig. 1b).

EFFECT OF THE MAGNETIC FIELD

In accordance with the experimental and theoretical results, external tilted magnetic field substantially affects the characteristics of electron transport in semiconductor nanostructures [11, 17, 19].

The effect of the magnetic field is taken into account in the semiclassical approximation in the calculation of the drift velocity of electrons versus longitudinal electric field $\upsilon(F)$. As was mentioned, dependence $\upsilon(F)$ obeys Esaki–Tsu formula (4) at low temperatures in the absence of magnetic field. Figure 2a shows such a dependence. It is seen that the drift velocity of carriers increases with increasing electric field and reaches a maximum that is known as the

[1]The calculations (including the below calculation of the spectrum of Lyapunov exponents) are performed in terms of dimensionless quantities. For convenience of the analysis of the results, several quantities on the plots are presented in dimensional units.

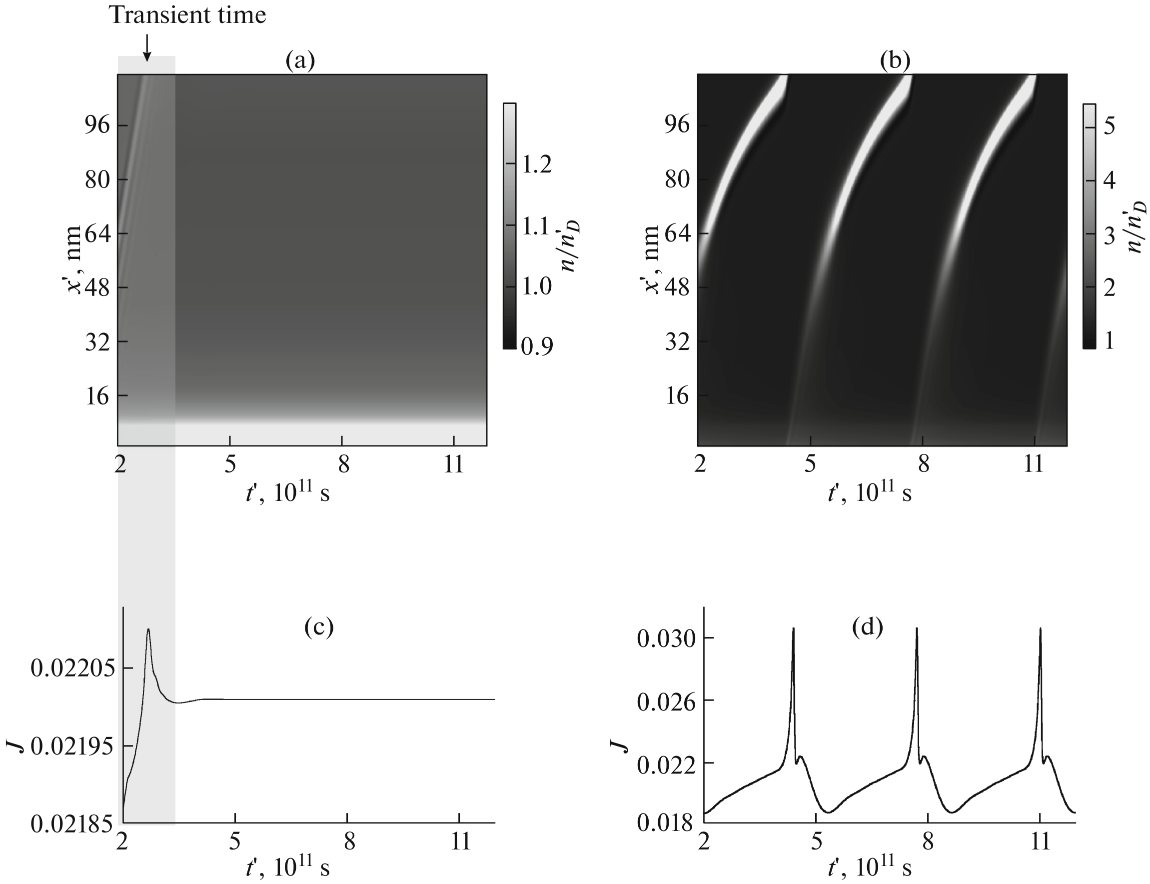

Fig. 1. Spatiotemporal dependences of (a, b) concentration of carriers and (c, d) current that flows through the superlattice for applied voltages V = (a, c) 9 and (b, d) 11 (V' = (a, c) 330 and (b, d) 400 mV).

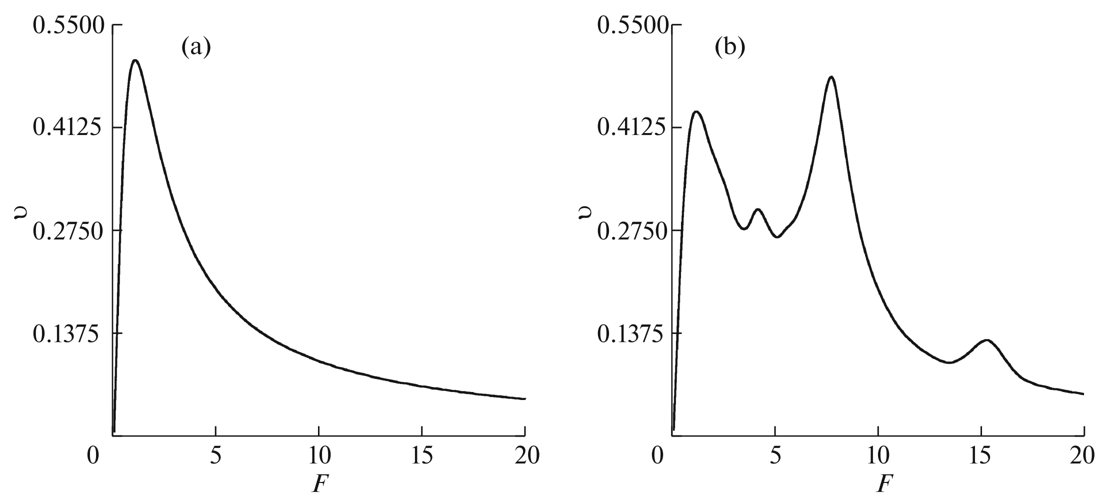

Fig. 2. Plots of the mean drift velocity of carriers in the first miniband of the semiconductor superlattice vs. electric field strength (a) in the absence and (b) in the presence of magnetic field at B = 15 T and Θ = 40°.

Esaki–Tsu peak. A further increase in electric field strength F leads to a decrease in the velocity due to the scattering of carriers and the Bragg reflection of electrons from the edges of the energy miniband. Such an effect has been demonstrated in [1] and is also related to the THz Bloch oscillations of electrons. The falling fragment of curve $\upsilon(F)$ determines the negative differential conductance of semiconductor nanostructures

[20] and gives rise to nonstationary regimes of electron transport.

External magnetic fields and temperature primarily affect the dynamics of carriers in the semiconductor superlattice and, hence, dependence $\upsilon(F)$. In this case, analytical calculations are impossible and the dependence must be numerically calculated [17]. In this work, we analyze the effect of the tilted magnetic field and calculate dependence $\upsilon(F)$ using the method of [17]. Figure 2b shows such a dependence for the magnetic field with induction $B' = 15$ T and tilt angle $\Theta = 40°$. It is seen that the magnetic field strongly affects the behavior of the drift velocity upon variation in the electric field strength in the superlattice. Note the presence of additional peaks related to the Bloch cyclotron resonances [17]. The amplitudes of such peaks can be greater than the amplitude of the Esaki–Tsu peak (Fig. 2b).

Thus, the magnetic field may significantly affect the properties of a semiconductor nanostructure and, hence, the stability of the stationary state therein and the parameters of the nonstationary dynamic regimes.

METHOD FOR CALCULATION OF THE LYAPUNOV EXPONENTS

Methods based on the Benettin algorithm [21] are normally used in the calculation of the Lyapunov exponents. Such an algorithm involves the simulation of the dynamics of small (linear) perturbations of the reference state and calculation of their norm versus time with the aid of orthogonalization and normalization procedures [21]. In this case, the reference state of the system is determined by a set of dynamic variables that unambiguously describe the state of the system at each time moment.

At each moment, the state of the semiconductor structure under study is determined by spatial distributions of electric field $F(x)$, concentration of carriers $n(x)$, and current density $J(x)$. A set of these quantities can be considered as the reference state. Following the approach of [22] in which the Lyapunov exponents are calculated for spatially distributed systems, we eliminate electric field strength and current density from the reference state, since both quantities are one-to-one determined by the concentration of carriers using Eqs. (1) and (3). Thus, the reference state of the system under study can be represented as quantity

$$U(x,t) = n(x,t), \tag{6}$$

which depends on time and spatial coordinate.

Spatially distributed perturbation $\tilde{n}(x, t)$ serves as the small perturbation of state (6). The normalization condition $\sqrt{(\tilde{n}(x,t_0)\tilde{n}(x,t_0))} = 1$ is satisfied at the initial moment, where scalar product $(\tilde{n}, \tilde{n})$ is written as

$$(\tilde{n},\tilde{n}) = \int_0^1 \tilde{n}(x)\tilde{n}(x)dx. \tag{7}$$

Time evolution of the perturbation is simulated using operator (1), (3)–(5) linearized in the vicinity of the reference state:

$$\begin{gathered}\frac{\partial \tilde{F}}{\partial x} = \nu\tilde{n},\\ \frac{\partial \tilde{n}}{\partial t} = -\beta\frac{\partial \tilde{J}}{\partial x},\\ \tilde{J} = \tilde{n}\upsilon(F) + n\frac{\partial\upsilon(\tilde{F})}{dF}\tilde{F},\\ \int_0^1 \tilde{F}(x)dx = 0.\end{gathered} \tag{8}$$

To calculate the spectrum of the Lyapunov exponents, we introduce a set of perturbations $\tilde{n}_i(x, t)$ that satisfies the orthogonality condition:

$$(\tilde{n}_i(x), \tilde{n}_j(x)) = \begin{cases}0, & i \neq j\\ 1, & i = j.\end{cases} \tag{9}$$

Such a set can be obtained with the aid of the Gram–Schmidt procedure, which is represented in the following way for spatially distributed systems [20]:

$$\begin{gathered}\hat{n}_1(x,t_{\rm GS}) = \varphi_1(x),\\ \hat{n}_{i+1}(x,t_{\rm GS})\\ = \varphi_{i+1}(x) - \sum_{k=1}^{i}(\hat{n}_k(x,t_{\rm GS}),\varphi_{i+1}(x))\hat{n}_k(x,t_{\rm GS}),\\ i = \overline{1, N-1},\\ \tilde{n}_i(x,t_{\rm GS}) = \frac{\hat{n}_i(x,t_{\rm GS})}{|\hat{n}_i(x,t_{\rm GS})|}, \quad i = \overline{1,N}.\end{gathered} \tag{10}$$

Here, $t_{\rm GS}$ is the time moment at which the procedure is employed.

We integrate Eqs. (1) and (3)–(5) simultaneously with Eqs. (8), which describe the dynamics of the set of perturbations, and periodically employ procedure (10) with recalculated functions $\varphi_i(x) = \tilde{n}_i(x, t_{\rm GS})$ to calculate the Lyapunov exponents as

$$\Lambda_i = \frac{1}{MT}\sum_{j=1}^{M}\ln(\sqrt{(\tilde{n}_i(x,jT),\tilde{n}_i(x,jT))}), \tag{11}$$

where i is the exponent number, M is the number of renormalization and orthogonalization operations (11), and T is the time interval between renormalizations. Figure 3 illustrates the calculation of four high-order Lyapunov exponents for $M = 6000$ and $T \sim 2.5$ ($T = 6.25 \times 10^{-13}$ s). It is seen that multiple ($M \sim 6000$) orthogonalizations and normalizations make it possible to obtain almost constant Λ that correspond to the Lyapunov exponents of the system.

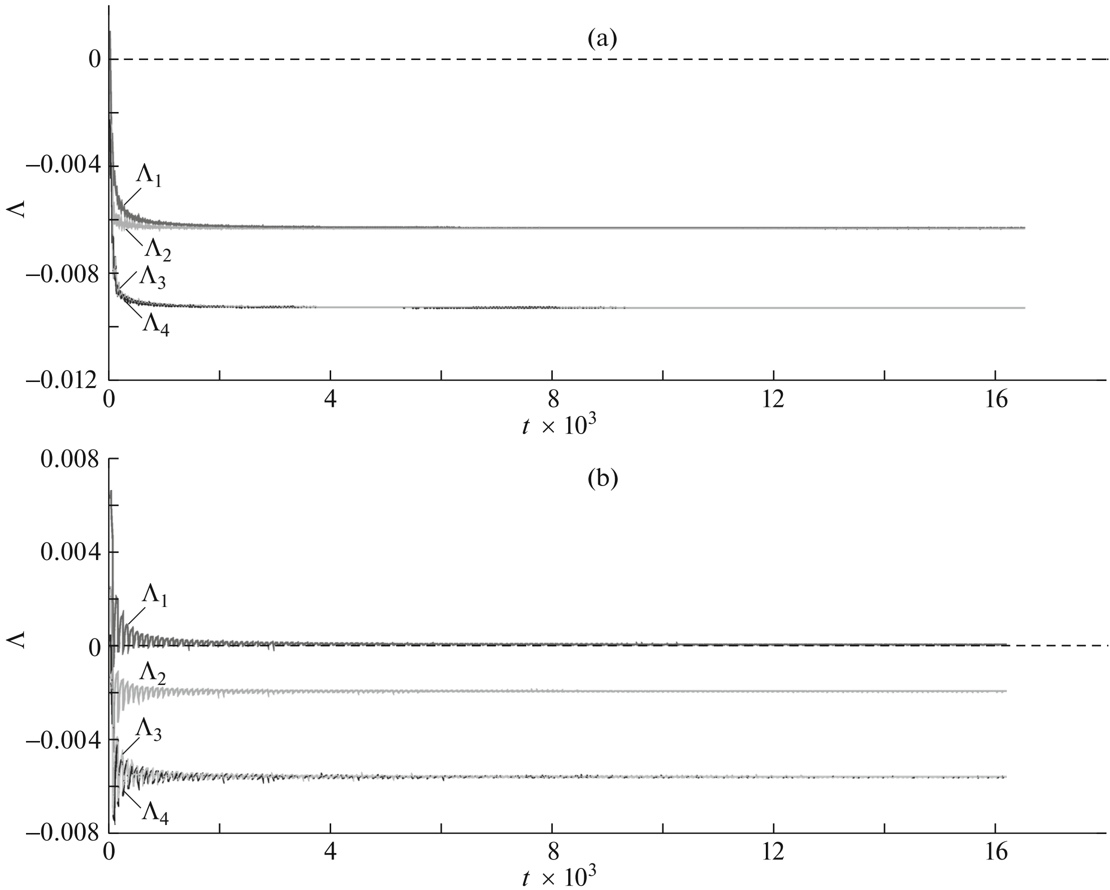

Fig. 3. Plots of four high-order Lyapunov exponents vs. time for an autonomous semiconductor superlattice in (a) stationary state $V \sim 9$ ($V' = 330$ mV) and (b) regime of periodic generation $V \sim 11$ ($V' = 400$ mV).

The Lyapunov exponents in Fig. 3 are calculated for $V \sim 9$ ($V' = 330$ mV) and $V \sim 11$ ($V' = 400$ mV) (Figs. 3a and 3b, respectively) for which Fig. 1 shows the numerically simulated dynamics of the reference state. It is seen that only negative high-order Lyapunov exponents correspond to the stationary state (Fig. 3a) whereas the regime of periodic generation is characterized by the zero high-order exponent (Fig. 3b).

The proposed method is used to calculate the dependences of five high-order Lyapunov exponents on the applied voltage for both autonomous dynamics (Fig. 4a) and in the presence of the tilted magnetic field (Fig. 4b). We take into account the effect of the magnetic field using dependences $\upsilon(F)$ and $d\upsilon(F)/dF$ in Eqs. (1), (3)–(5), and (8), which describe the evolution of the reference state and the set of perturbations. In both cases, zero positive Lyapunov exponent appears in the spectrum when the voltage increases. Such a results indicates a transition to the nonstationary dynamics and the generation of the oscillations of current that flows through the structure. The magnetic field affects the voltage that is needed for the development of generation (the voltage increases to 560 mV). In spite of the effect of the magnetic field that causes additional falling fragments on dependence $\upsilon(F)$ (Fig. 2b), the oscillations in the system remain periodic at relatively high applied voltage (zero high-order Lyapunov exponent is obtained).

Thus, the method proposed for the calculation of the spectrum of the Lyapunov exponents is efficient in the analysis of the stability of the stationary state in the semiconductor superlattice and identification of the type of oscillation dynamics. The method is also employed for the analysis of the effect of the tilted magnetic field on the dynamics of the system in the stationary state and generation regime.

CONCLUSIONS

We have proposed a method to calculate the spectrum of the Lyapunov exponents for a semiconductor superlattice that is described in the framework of the semiclassical approach using partial differential equations. The method is used to analyze the dynamics of an autonomous superlattice and the dynamics of the system in the presence of the tilted magnetic field. When dc voltage is applied, the system exhibits instability in both cases and periodic oscillations are gener-

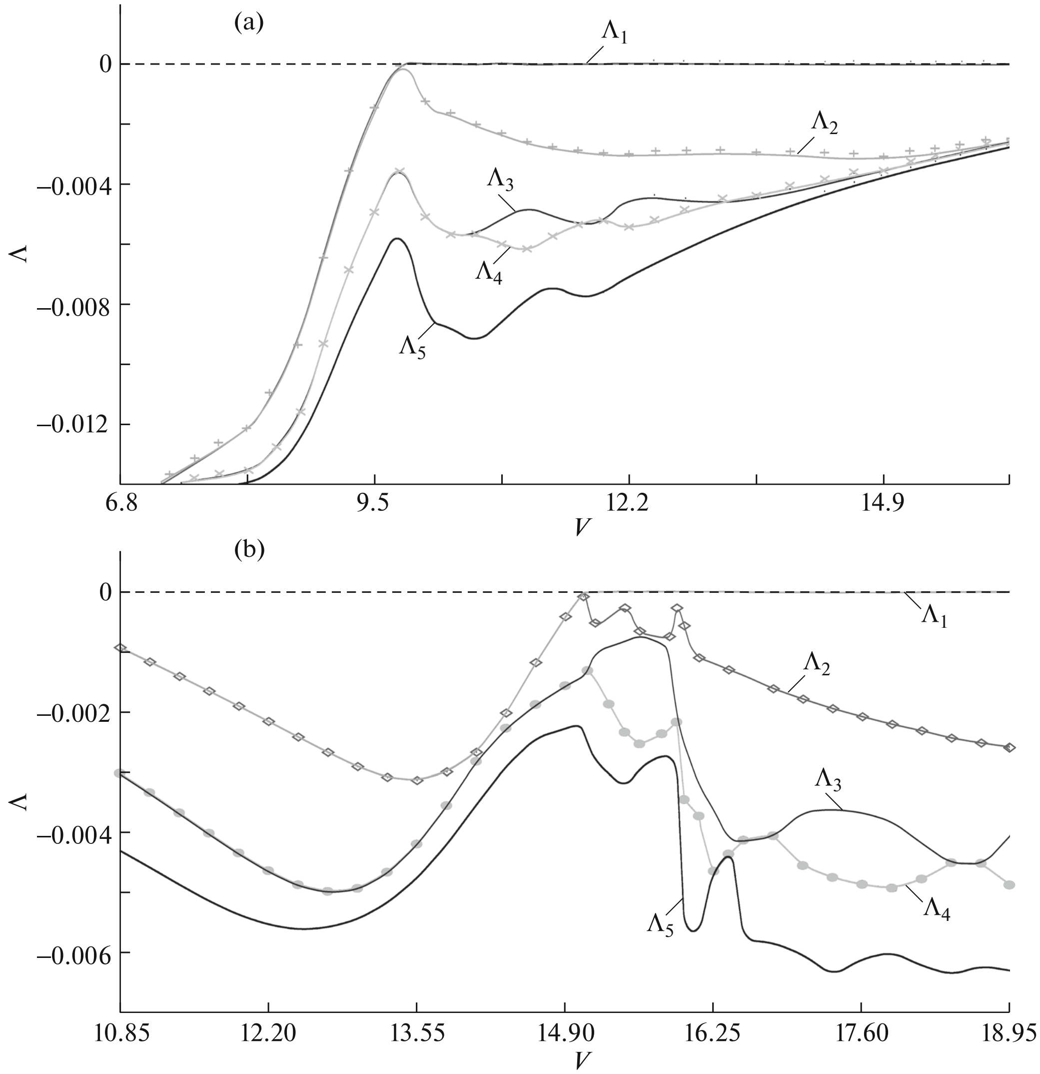

Fig. 4. Plots of five high-order Lyapunov exponents vs. dimensional applied voltage (a) for autonomous dynamics and (b) in the presence of the tilted magnetic field.

ated. However, the voltages needed for the instability are significantly different. The analysis of the spectrum of the Lyapunov exponents makes it possible to study the stability of the system and identify the type of nonstationary dynamics when the voltage increases. In the autonomous semiconductor system, an increase in the applied voltage does not lead to changes of the type of dynamics and the oscillations remain periodic. Variations in the magnetic field also do not cause changes of the type of dynamics, which is proven by a zero high-order Lyapunov exponent for a wide range of applied voltages.

The method proposed for the calculation of the spectrum of the Lyapunov exponents is efficient in the analysis of the stability of the stationary state and identification of the type of oscillation regime both for autonomous dynamics and in the presence of a magnetic field. The method is promising for the analysis of complicated regimes of electron transport in semiconductor nanostructures that interact with external electrodynamic structures and fields, which is important for the simulation of real sub-THz and THz electronic devices.

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REFERENCES

1. L. Esaki and R. Tsu, IBM J. Res. Dev. **14** (1), 61 (1970).

2. M. I. Ovsyannikov, et al., Sov. Phys. Semicond. **4**, 1919 (1970).
3. A. Ya. Shik, Sov. Phys. Semicond. **8**, 1195 (1974).
4. L. L. Bonilla and H. T. Grahn, Rep. Prog. Phys. **68**, 577 (2005).
5. T. M. Greenaway, et al., Phys. Rev. B **80**, 205 318 (2009).
6. T. Hyart, et al., Phys. Rev. Lett. **103**, 117 401 (2010).
7. B. K. Ridley, Proc. Phys. Soc. London **82**, 954 (1963).
8. A. A. Koronovskii, et al., Phys. Wave Phenom. **21** (1), 48 (2013).
9. A. Sibille, et al., Phys. Rev. Lett. **64**, 52 (1990).
10. H. Kroemer, IEEE Trans. Electron Devices **15**, 819 (1968).
11. A. G. Balanov, et al., Phys. Rev. E **77**, 026 209 (2008).
12. S. P. Kuznetsov, *Dynamic Chaos,* Ser. *Modern Theory of Waves and Oscillations* (Fizmatlit, Moscow, 2001).
13. V. V. Makarov, et al., Izv. Vyssh. Uchebn. Zaved., Prikl. Nelineinaya Din. **21** (5), 40 (2013).
14. S. Lepri, et al., Chaos **7**, 701 (1997).
15. S. P. Kuznetsov and D. I. Trubetskov, Radiophys. Quantum Electron. **47** 341 (2004).
16. T. M. Fromhold, et al., Nature **428**, 726 (2004).
17. A. O. Selskii, et al., Phys. Rev. B **84**, 235 311 (2011).
18. A. E. Hramov, et al., Phys. Rev. Lett. **112**, 116 603 (2014).
19. A. O. Sel'skii and G. V. Osipov, Tech. Phys. Lett. **41**, 98 (2015).
20. M. Butiker and H. Thomas, Phys. Rev. Lett. **38** (2), 78 (1977).
21. G. Benettin, et al., Meccanica **15**, 9 (1980).
22. A. A. Koronovskii, et al., Phys. Rev. B **88**, 165 304 (2013).

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