Analysis of the establishment of the global synchronization in complex networks with different topologies of links

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ABSTRACT

In the present paper the mechanism of the global synchronization onset through the formation of the synchronous clusters in complex networks with different topologies of links (scale-free networks, small-world networks, random networks) is studied. We consider the dependencies of integral characteristics of synchronous dynamics (synchronization measure, number of synchronous clusters, etc) on coupling strength between nodes. As a basic element of the node oscillator we consider Kuramoto phase oscillator.

Keywords: complex networks, scale-free networks, small-world networks, random networks, synchronization, clusters, Kuramoto oscillator

1. INTRODUCTION

Global processes in the modern world inevitably lead to the establishment of global connections between elements (subsystems), which results in formation and evolution of various network structures. Networks is all around us and we ourselves are also objects of social and socio-economic networks. On the one hand, networks can be observable physical objects, such as electric power grids, the Internet, highways or subway systems, neural networks. On the other hand, networks can be defined in the abstract area, for example, the network of aviation traffic or social networks where collaborations between individuals being treated as relations of network elements.

Interactions between subsystems in the network may be described in terms of graphs. This approach was initiated by Erdös and Rényi (ER).[1] Recently, a small-world network was introduced by Watts and Strogatz (WS),[2] where a some of edges on a regular lattice is rewired with probability p to other nodes of the network. Moreover, there is another type of networks called scale-free (SF) networks,[3–7] which are ubiquitous in real-world networks such as the World Wide Web,[8,9] the Internet,[10,11] the citation network,[12] the author collaboration network of scientific papers,[13–15] and the protein interaction networks.[16] The peculiarity of such networks is that the degree distribution follows a power law $P(k) \sim k^{-\gamma}$ with $\gamma = 3$ for the BA model,[3] while for the ER and WS models, it follows a Poisson distribution[17] with the exception of random networks such as the actor network whose degree distribution follows a power law but has a sharp cutoff in its tail.[18]

In recent years, synchronization phenomena in the complex networks is actively studied by many researchers. There are many works devoted to the study of mechanisms of synchronization onset in networks with different types of topology.[19–23] It should be noted, that studies of synchronization have mostly been aimed at detection of influences of structural properties of small-world (SW),[24–26] as well as scale-free (SF) network.[22,27] Many researchers in their studies of synchronization in complex networks operate for the most part with one of the important characteristic for synchronization processes, namely the dependence of order parameter on coupling strength. The main purpose of our study was to find out some peculiarities being intrinsic for different types of network topology in synchronization onset from the viewpoint of synchronous clusters appearing. The method of cluster extraction was developed and used to analyse the dynamics of several networks with different types of relation topology.

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2. NUMERICAL MODEL AND APPLIED METHODS

Kuramoto model is one of the simplest and the most successful models in studying the synchronization. In the present work the network consisting of 500 Kuramoto oscillators with complex topology of relations between nodes was considered. The individual dynamics of each node is given by the differential equation

$$\dot{\varphi}_i = \omega_i + \lambda \sum_{j \in N_i} \varepsilon_{ij} \sin(\varphi_j - \varphi_i), \quad (1)$$

where ω_i are the natural frequencies, λ is the coupling parameter, N is the size of ensemble. The natural frequencies of the oscillators were distributed randomly in the range of 0.5 to 1.5. ε_{ij} are the elements of an $N \times N$ adjacency matrix, so that $\varepsilon_{ij} = 1$ when the oscillators i and j are connected while $\varepsilon_{ij} = 0$ otherwise. Sum of ε_{ij} gives rise to the degree k_i of the node the oscillator i corresponds to. To produce a random network following to ER model,[1] each two nodes of an ensemble consisting from N oscillators are connected with the probability p. The SW network considered in this paper has been constructed according to the work by D.J. Watts and S.H. Strogatz.[2] First, a ring of N nodes is constructed, with only nearest neighbor connections between the nodes. Then each local link is once removed and reconnected to a randomly chosen node with the rewiring probability p. In our study, static scale-free networks whose degree distribution $P(k)$ follows a power law have been also considered. To produce the SF network we use the procedure described in the work by K.-I. Goh, et al.[28] At the beginning there are N nodes in the ensemble. We assign the weight $p_i = i^{-\alpha}$ to each node, where i corresponds to the position of node in the network, α is a control parameter in $[0, 1)$. Next, we select two different nodes (i, j) with probabilities equal to the normalized weights, $p_i / \sum_k p_k$ and $p_j / \sum_k p_k$, respectively, and add a link between them unless one exists already. This process is repeated until mN edges are made in the system. Then the mean degree is $2m$. For all topologies described above connections were mutual, so multi-connections and self-connections were excluded during the process.

To extract clusters of oscillators demonstrating synchronous dynamics the method based on the analysis of the phase difference between interacting Kuramoto oscillators was developed. The idea of the method is the following. During numerical calculations after some period of time sufficient for dynamics to be established the phase difference $\Delta\varphi_1(i, j)$ of every pair of oscillators in the ensemble is calculated. Thus, the phase of every subsystem compares with phases of all other oscillators in the network. Then, after many steps, when changes in system dynamics is observable the same phase difference calculation is repeated. Next, for every couple of oscillators calculated phase differences, $\Delta\varphi_1(i, j)$ and $\Delta\varphi_2(i, j)$ are compared with each other. If the difference between these values more than 2π, one can confirm that oscillators with numbers i and j are nonsynchronized with each other, because an absolute value of their phase difference increases in time. Otherwise, when the phase difference remains limited in time, the oscillators demonstrate synchronous behaviour. Therefore, proposed method of synchronous cluster extraction allows identifying precisely nodes being synchronized with each other, as well as subsystems demonstrating asynchronous dynamics relative to other oscillators in ensemble.

3. RESULTS OF NUMERICAL SIMULATIONS

Fig. 1 and Fig. 2 show the dependencies of the size N_c of maximal synchronous cluster on coupling parameter λ for random network and SW network, correspondingly, with different values of probability p, being the same for every type of network topology. It should be noted, that the variation of the size of maximal synchronous cluster under changes of coupling parameter is very similar for both types of topology of links between nodes in network. The more probability value and, hence, degree of the node is, the earlier synchronous regime in the network appears. For SW network with small values of probability p the curve in figure is indented due to relatively small amount of global links in comparison with the greater values of probability p, which influences on the character of the whole system behaviour and affects the onset of synchronous dynamics in ensemble. Analogous results have been obtained for scale-free network with different values of control parameter α determining the distribution of edges between subsystems in the network.

In addition, the dynamics of several random networks of Kuramoto oscillators with different numbers of nodes but with equal node degree has been investigated from the viewpoint of the cluster formation with the increase of the coupling strength between subsystems. The dependencies of the relative size of synchronous

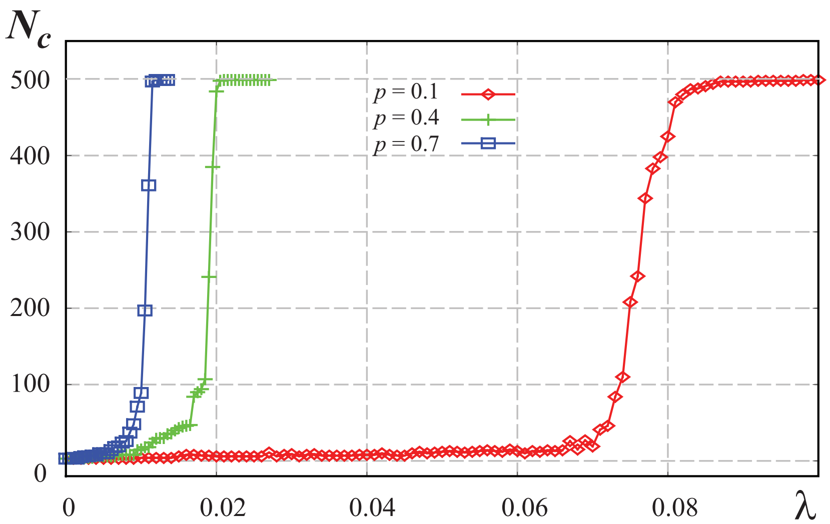


Figure 1. Dependence of the size N_c of maximal synchronous cluster on coupling parameter λ for random network consisting of 500 Kuramoto oscillators with different values of probability p.

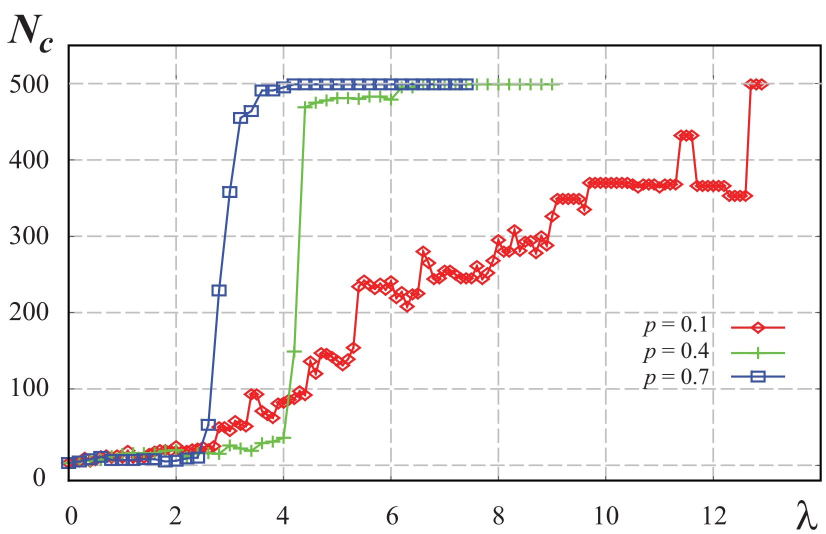
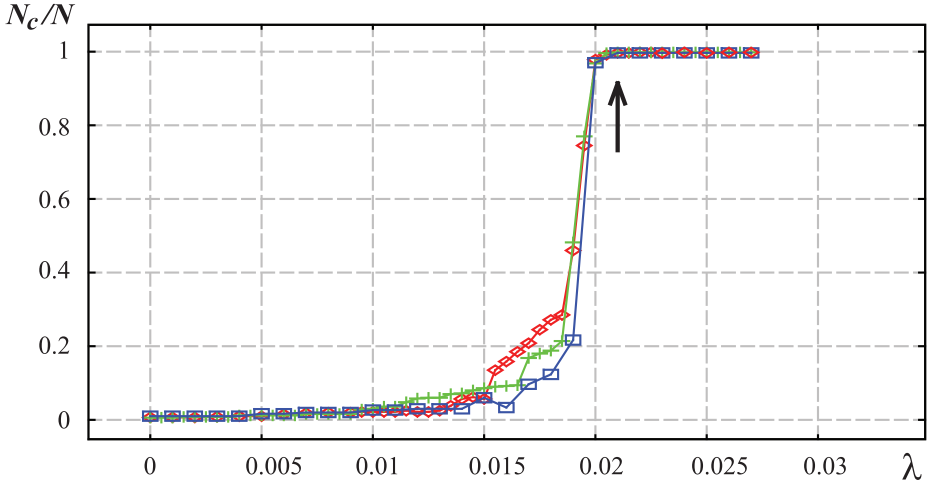


Figure 2. Dependence of the size N_c of maximal synchronous cluster on coupling parameter λ for small-world network consisting of 500 Kuramoto oscillators with different values of probability p.

cluster (the size of maximal cluster being normalized on the total number of oscillators in ensemble) on the coupling parameter between subsystems are very similar for the considered networks consisting of 600, 500 and 300 nodes. For all networks the mean node degree is equal to 200, whereas the values of probability p differ from one another. Fig 3(a) illustrates these dependencies. An arrow shows the critical value of coupling parameter, when all oscillators in the ensemble start demonstrating synchronous dynamics. It is important to note that the critical value of coupling parameter, when the size of the maximal cluster becomes equal to the total number N of oscillators in ensemble, i.e. all oscillators in the network are synchronized with each other, is the same for all considered networks. To verify the obtained results in Fig. 3(b) the dependence of the order parameter on the coupling strength is shown for the same networks consisting of 600, 500 and 300 nodes and the same values of probability p used for constructing edges between oscillators in the network. It could be seen clearly, that both characteristics, order parameter and size of maximal cluster demonstrate analogous behaviour under change of coupling parameter value.

Found relation can be explained by the fact that synchronizability of the random network is determined by



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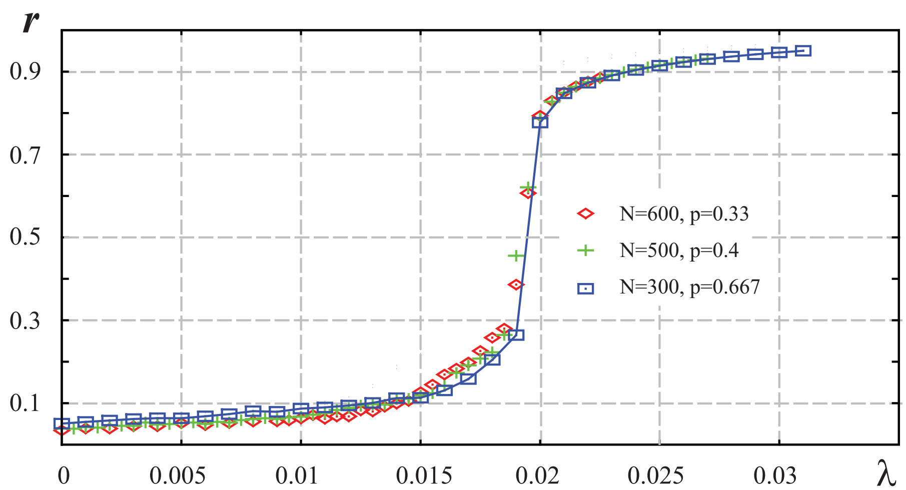

b

Figure 3. (a) Dependence of the size N_c of maximal synchronous cluster normalized on total number N of oscillators in the ensemble on coupling parameter λ. Red curve corresponds to the data for network of 600 Kuramoto oscillators ($p = 0.33$), green curve is for ensemble consisting of 500 nodes($p = 0.4$), and blue one is for the network of 300 oscillators ($p = 0.667$). For all networks node degree is equal to 200. An arrow shows the critical value of coupling parameter, when all oscillators in the ensemble start demonstrating synchronous dynamics.
(b) Dependence of the order parameter r on the coupling parameter value λ between oscillators in the ensemble. The product of network size N and probability p is equal to 200.

the intensity of links and node degree (average number of links for one node). In other words, it depends on dimensionless complex being a product of coupling parameter and node degree or that is the same, a product of coupling parameter, probability p and number of nodes N in the network. The more the value of probability is, the more the node degree is, and, hence, the less critical value of the coupling parameter corresponding to the onset of synchronous regime in the network of Kuramoto oscillators is. Therefore, the increase of probability p results in decrease of the critical value of coupling strength inversely proportional to the probability value. Dependence of the threshold coupling parameter on probability p is shown in Fig. 4 for the random network consisting of 500 oscillators. An analytical approximation is shown by the solid line, while points correspond to

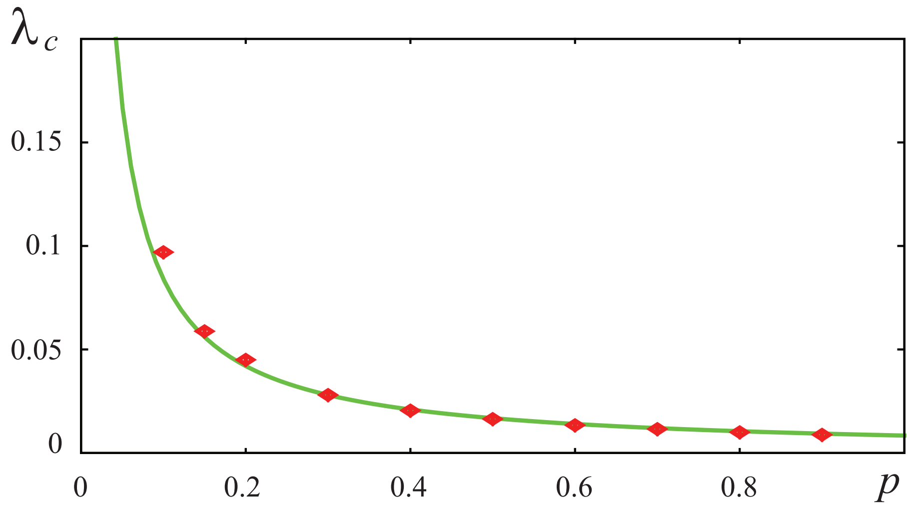

Figure 4. Dependence of the threshold value of coupling parameter λ_c on probability p for the random network consisting of 500 Kuramoto oscillators. Points correspond to the data obtained numerically, solid lines illustrates an analytical approximation $\lambda_c = 0.0084/p$.

the numerical data. It can be seen clearly, that this dependence is inverse proportionality, as it was described above.

Obtained results also lead to the following conclusion. Considering two networks with different number of nodes with probability of connection every pair of oscillators has been selected in such a way that a product of number of nodes in the network and the probability value is equal for both networks, the dependence of order parameter on coupling strength should also be the same. In particular, the threshold of synchronous regime appearance should be identical despite the different sizes of networks.

4. CONCLUSIONS

In this paper we report on the results of the study of mechanism of the global synchronization onset through the formation of the synchronous clusters in complex networks with different topologies of links (small-world networks, random networks, scale-free networks). Method for synchronous clusters detection has been developed and used for analysis of dynamics of Kuramoto oscillators network with different types of topology of relations between subsystems and different number of nodes. Dependencies of the size N_c of maximal synchronous cluster on coupling parameter λ for networks consisting of 500 Kuramoto oscillators with different values of probability p were obtain. It has been also shown that random networks with different numbers of nodes but equal node degree have the same critical value of coupling parameter corresponding to the moment when all oscillators in the ensemble start demonstrating synchronous dynamics. This threshold value of coupling parameter is inversely proportional to the probability of establishing relation between each two nodes when constructing the network.

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