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Establishing Generalized Synchronization in Rössler Oscillator Networks

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Abstract—The processes of establishing of generalized chaotic synchronization in a network of mutually coupled continuous-time systems are investigated. The nature of interaction between the network elements in transitioning from synchronous to asynchronous behavior while increasing the communication parameter is studied. A synchronization regime, the nearest neighbors method, and calculations of the spectrum of Lyapunov exponents are used to clarify features of the interaction between network elements and the occurrence of a generalized chaotic.

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INTRODUCTION

The study of different types of chaotic synchronization (e.g., phase synchronization, lag synchronization, and total synchronization) is of considerable interest to researchers [1]. Among the different types of synchronous behavior, the study of such phenomena as generalized chaotic synchronization arouse great interest [2–4]. The phenomenon of generalized synchronization can occur between different oscillators, including those between oscillators with different phase space dimensions. A criterion for the establishing of a generalized synchronization regime is the existence of a functional dependence between the states of interacting oscillators. Systems with discrete time coupled unidirectionally and mutually [5], and flow systems with unidirectional [6] and mutual [7] connections (particularly spatially distributed systems [8–10]) have been investigated in the past.

The next step in studying the phenomenon of generalized synchronization was to investigate networks of nonlinear elements. The process of establishing a generalized synchronization regime was studied by changing from asynchronous behavior to synchronous in a small network of logistic maps [7]. It should be noted that because of the complexity of the considered systems [8], there many questions remain in this line of research. This work considers the problem of establishing a regime of generalized chaotic synchronization in a network of mutually connected flow systems.

MODEL SYSTEM AND RESEARCH METHODS

A network consisting of $N = 5$ mutually connected Rössler oscillators was selected as our model network of mutually connected flow systems. The evolution of the i-th ($i = 1,\dots,N$) network element is described by the system of equations

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon\sum_{j=1}^{N} C_{ij}x_j, \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= p + z_i(x_i - c), \end{aligned} \quad (1)$$

where $a = 0.15$, $p = 0.2$, and $c = 10$ are control parameters; ε is the coupling coefficient; C_{ij} is the communication matrix ($C_{ij} = 0$, there is no connection between elements; $C_{ij} = 1$, each element interacts with the other elements of the network); $\omega_1 = 0.95$, $\omega_2 = 0.9525$, $\omega_3 = 0.955$, $\omega_4 = 0.9575$, $\omega_5 = 0.96$. The each-to-each type of connection between network elements was used (i.e. $C_{ij} = 1$).

The auxiliary system method [5] is a classic means of detecting generalized synchronization regimes in unidirectional coupled systems, but this method cannot be used to study the interaction of mutually coupled systems, for the reasons described in [8, 12, 13]. We therefore need to use other methods, e.g., calculating the spectrum of Lyapunov exponents [12, 14]. In considering two coupled oscillators, the transition via zero to the region of negative values of one of the two largest Lyapunov exponents is a criterion for the existence of generalized synchronization. The other large Lyapunov exponent remains positive [6, 15] (a situation with two positive Lyapunov exponents of autonomous oscillators, i.e., the hyperchaotic regime of oscillations, is not considered in this work). In a network consisting of N nonlinear elements, the spectrum of Lyapunov exponents is characterized by N positive values at low values of the coupling coefficient (which, as the coupling coefficient grows, will continue to

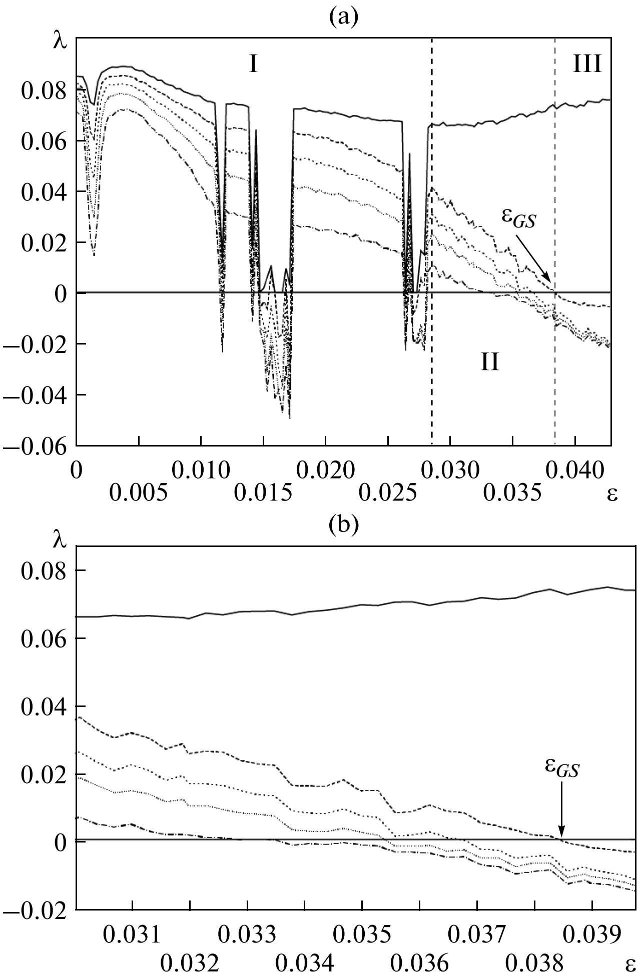

Fig. 1. (a) Dependence of the five Lyapunov exponents on coupling coefficient ε and (b) domain II of values on a larger scale.

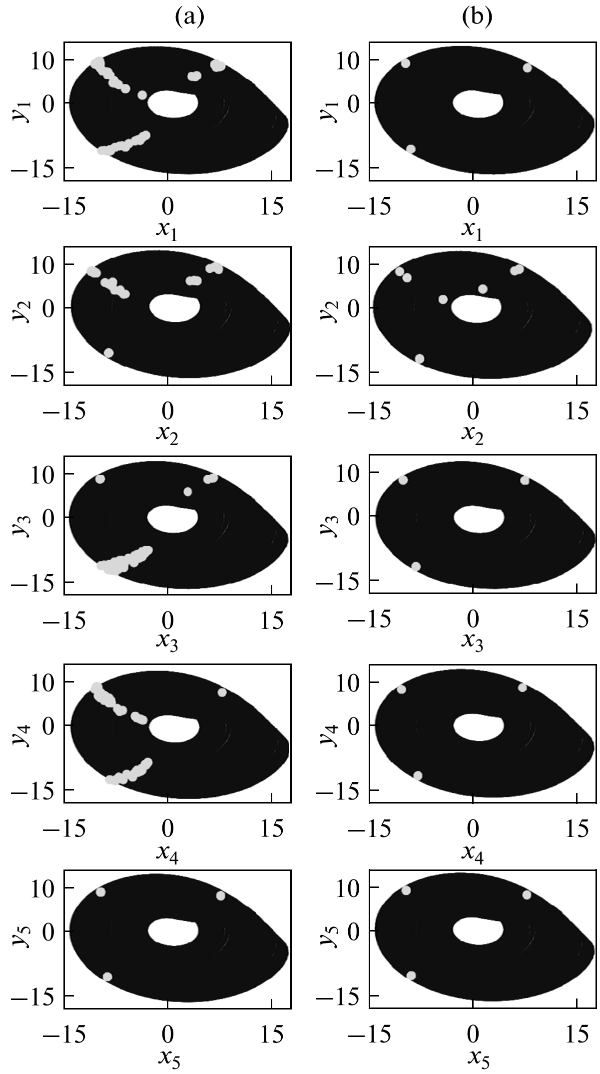

Fig. 2. Phase portraits of the five considered oscillators for two values of the coupling coefficient: (a) $\varepsilon_1 = 0.0345$ and (b) $\varepsilon_2 = 0.038$

transition to the region of negative values [8]). For a network of five coupled Rössler oscillators, the transition to the region of negative values for four out of the five Lyapunov exponents (the first exponent remains positive) serves as a criterion for the existence of a generalized synchronization regime in the system.

Fig. 1a presents the dependence of five Lyapunov exponents for system of mutually coupled Rössler oscillators (1) on the value of coupling parameter ε. Dependency $\lambda_i(\varepsilon)$ ($i = 1, ..., 5$) can be divided into three domains (Fig. 1a): (I) $\varepsilon \in [0;0.032]$ is the domain of asynchronous behavior; (II) $\varepsilon \in [0.032;0.0385]$ is the region of the transition to the regime of generalized synchronization; (III) is the domain corresponding to the regime of generalized chaotic synchronization. In the domain of the transition to the regime of generalized chaotic synchronization, the four Lyapunov exponents consecutively change sign in crossing the x-axis (Fig. 1b). The time of the sign change for the Lyapunov exponent second in priority (i.e., the point of the final transition to the generalized synchronization regime) with coupling coefficient $\varepsilon = 0.0385$ is marked by an arrow in Fig. 1b. At the borders of domain II, the system thus demonstrates both asynchronous ($\varepsilon < 0.032$) and synchronous behavior ($\varepsilon > 0.0385$). At the same time, in contrast to the case of two oscillators, the border between asynchronous and synchronous behavior is in this case blurred, and there is a gradual restructuring of the behavior of the network elements, the basic mechanisms and laws of which are still unknown.

The nearest neighbor method was used to determine details of the processes occurring in domain II, where there was a gradual sequential change in sign for the Lyapunov exponents [8]. In this method, when considering the behavior of coupled oscillators in one

Table 1. Values of variance for coupling coefficient $\varepsilon_1 = 0.0345$

	x_1	x_2	x_3	x_4	x_5
x_1	–	5.57E–01	3.66E–02	4.82E+00	1.22E+00
x_2	4.94E+00	–	2.72E–01	3.54E+00	4.18E+00
x_3	6.62E–01	4.72E–02	–	8.11E–01	5.65E–02
x_4	2.06E+00	9.97E–01	9.50E–01	–	7.03E–01
x_5	4.19E–05	2.92E–05	6.95E–03	4.37E–04	–

Table 2. Values of variance for coupling coefficient $\varepsilon_2 = 0.038$

	x_1	x_2	x_3	x_4	x_5
x_1	–	5.20E–05	3.21E–04	2.78E–04	1.83E–04
x_2	7.64E–02	–	6.23E–02	2.58E–01	2.91E–02
x_3	3.09E–05	3.10E–05	–	3.10E–05	3.13E–05
x_4	2.67E–05	3.40E–05	5.45E–05	–	2.01E–05
x_5	6.94E–05	9.14E–05	1.39E–04	9.14E–05	–

neighbor, we must select reference point x_n^l and find selected reference point x_j^l for the nearest neighbors. We then need to find the maps of nearest neighbors x_j^m in the phase spaces of other oscillators [8] and consider the magnitude of the dispersion of distances between images of nearest neighbors x_j^m as a numerical characteristic of synchronism.

SIMULATION RESULTS

In this work, two values of the coupling coefficient, $\varepsilon_1 = 0.0345$ and $\varepsilon_2 = 0.038$, were chosen so that only one Lyapunov exponent was negative at ε_1, while three exponents were negative at the value of coupling coefficient ε_2 (Fig. 1b). Figures 2a,b show the phase portraits of the five interacting Rössler oscillators of the considered network for two values of the coupling coefficient, ε_1 and ε_2, respectively. Three reference points (one for each system) were selected on the phase portraits of three oscillators $i = 2,..,4$. In accordance with the method described above, their nearest neighbors were then found along with the corresponding maps in all other coupled systems. It can be seen that at $\varepsilon_1 = 0.0345$ (Fig. 2a), the points on the phase portraits of four out of the five oscillators are concentrated in the restricted domain of the attractor and distributed along its radius, while on the phase portrait of fifth oscillator $i = 5$, the maps of the nearest neighbors are close to one another and no blurring of states is observed. At the same time, on the phase portraits of all oscillators (with the exception of the second oscillator, $i = 2$) the maps of nearest neighbors have close values when coupling coefficient $\varepsilon_2 = 0.038$. These observations are confirmed by the values listed in Tables 1 and 2 for the variance of distances between nearest neighbors of the considered network node. The column numbers correspond to the numbers of the nodes on which reference point x_n^l was detected, while the line numbers correspond to the numbers of those nodes on which the nature of interaction was revealed.

In considering obtained values, it is easy to see that the magnitudes of variance for some pairs of network nodes differ considerably from the others. In accordance with the method of the nearest neighbors, we may conclude that at a low magnitude of variance (in this case, $\sim 10^{-4}$), the considered elements are synchronized in terms of generalized synchronization, while with greater variance ($\sim 10^{-1}$) the elements are not yet in the synchronism regime. We may therefore conclude that when coupling coefficient $\varepsilon_1 = 0.0345$, four of the five Rössler oscillators display synchronous (in the sense of generalized chaotic synchronization) behavior. The opposite situation is observed at the other end of domain II when coupling coefficient $\varepsilon_2 = 0.038$,: only the second system displays asynchronous behavior while the other elements of the network are synchronized with one another.

CONCLUSIONS

The phenomenon of generalized synchronization in a network of coupled mutually Rössler oscillators was investigated. The nature of interaction between nonlinear network elements in the domain of the transition from asynchronous to synchronous behavior was determined using calculations of the Lyapunov exponent spectrum and the nearest neighbor method.

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