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# Analyzing Cluster Formation in Adaptive Networks of Kuramoto Oscillators by Means of Integral Signals

**A. A. Harchenko[a], V. V. Makarov[a, b], and A. E. Hramov[a, b]**

[a] *Faculty of Nonlinear Processes, Saratov State University, Saratov, 410012 Russia*
[b] *NOTs Nonlinear Dynamics of Complex Systems, Saratov State Technical University, Saratov, 410054 Russia*
*e-mail: ainadil@mail.ru; hramovae@gmail.com*

**Abstract**—A numerical study of an adaptive network of coupled oscillators (Kuramoto oscillators) is performed. The problem of studying phase synchronization in networks by considering wavelet spectra of the integral signal and the evolution of the phase difference in clusters of the adaptive network is examined. The process behind the formation of phase clusters is analyzed using integral characteristics.

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## INTRODUCTION

One of the most important problems of modern radiophysics when studying oscillator networks is investigating the synchronization modes of interacting elements [1, 2]. The nodes in such networks serve as components of these complex systems, and the bonds between the nodes represent the interaction between them. There is now interest in studying networks whose topology is developed and changes over time, due either to external impacts or predetermined rules of evolution [3]. Such networks are called adaptive networks, and their study is of great interest both in terms of the fundamental issues of nonlinear dynamics and for applications in various branches of natural science, along with studying biological, social, economical and other systems that consist of large numbers of agents with different types and intensities of mutual coupling [4–7].

Such studies are especially important for neurodynamics and neurophysiology, where the objects of study are the neural networks of the central and peripheral nervous systems and complex network elements with their own complicated dynamics—neurons with constantly shifting and changing bonds, depending on the problem [8]. One traditional and very effective way of studying the electrical activity of the brain is the registration of electroencephalograms (EEGs), the average sum of the electric fields generated by the synaptic currents of a large group of neurons in the vicinity of the recording electrode [9]. With animals, implanted electrodes are traditionally used, allowing us to obtain more detailed information about the electrical activity of relatively small populations of neurons in cerebral cortexes and subcortical structures. Note that in both cases, the electroencephalogram signals are the averaged (integral) characteristics describing the dynamics of a complex neural network. An increase in amplitude in an EEG indicates the growing coherence of oscillations in the ensemble of neurons near the recording electrode.

Problems related to studying synchronous modes in neural networks of the brain are now of great interest when investigating pathological activity, particularly epileptic activity. An important problem is thus assessing the effectiveness of using integral parameters representing ensemble-averaged characteristics for this purpose [10, 11].

The aim of this work was to study a network of phase Kuramoto oscillators as a classic base model in the theory of networks [3], and to analyze the emergence of clusters according to their integral characteristics. As our object of study, we consider a model of a complex network with an adaptive communication system in which synchronous behavior leads to the emergence of clusters of interacting adjacent elements. The dynamics of this network is analyzed using a continuous wavelet transform of integral network characteristics [10, 11], allowing us to perform diagnostics of the formation and dynamics of clusters. Using the phase distribution enables us to quantify the level of clustering in the model network.

## MODEL AND DIAGNOSTICS OF NETWORK SYNCHRONIZATION

In this work, we use the model of a phase Kuramoto oscillator, one of the most widely used basic models in the theory of networks, which serves as a mathematical interpretation of the collective dynamics of chemical and biological oscillators [12]. Different modifications of this model network of phase oscillators are used extensively in analyzing the processes of clustering and synchronization, including neural networks and social systems [3].

Our basic model is a network of coupled Kuramoto oscillators in which each node of the network has coupling $\omega_{ij}$ with other nodes and phase $\varphi_i$ that vary over time:

$$\dot{\varphi}_i = \delta_i + \lambda \sum_{j=1}^{N} \omega_{ij} \sin(\varphi_j - \varphi_i), \qquad (1)$$

Here, $\delta_i$ represents the random angular frequencies of the Kuramoto oscillators, $N$ is the number of oscillators in the network, $\lambda$ is the intensity of coupling between the oscillators, and $\omega_{ij}$ is the weight of coupling between nodes $j$ and $i$.

In this work, we use the model proposed in [3] for a complex network with adaptive coupling. This model reflects two key features of natural networks: a scaleless distribution of the coupling strength and the formation of mesoscale structures. These phenomena can lead to the emergence of two mechanisms: hemophilia, associated with the strengthening of links between synchronized nodes, and homeostasis, a mechanism of competition by means of which an increase in the coupling of one element of a network is balanced by a weakening of the others in the same node under the condition

$$\sum_{j \neq i}^{N} \omega_{ij} = 1, \qquad (2)$$

In accordance with (2), the sum of all weights in the node is constant at any moment in time, and $\omega_{ij}$ is a coefficient describing the strength of coupling connecting nodes $j$ and networks $i$.

It varies over time according to the law [3]

$$\dot{\omega}_{ij}(t) = \omega_{ij}(t)\left[s_i p_{ij}^T(t) - \sum_{l=1}^{N} \omega_{il} p_{ij}^T(t)\right], \qquad (3)$$

where $s_i = \sum_{j=1}^{N} \omega_{ij}$ is the total incoming force of node $i$, and $p_{ij}^T(t)$ is the degree of local synchronization between oscillators $i$ and $j$, averaged over time in interval $[t-T, t]$. It is given by the equation

$$p_{ij}^T(t) = \left|\frac{1}{T}\int_{t-T}^{t} e^{\sqrt{-1}[\varphi_j(\tau) - \varphi_i(\tau)]} d\tau\right|. \qquad (4)$$

Here as in [3, 13], control parameter $T$ was set at $T = 100$ for all calculations.

The possibility of detecting clusters in adaptive networks based on a wavelet analysis of the macroscopic dynamics was established in [11, 14]. For our analysis of synchronization using integral characteristics, we consider an integral signal representing averaged fluctuations within a certain subset of $N$ elements of the network (in this work, we consider the signal obtained by averaging over the network of $N$ oscillators)

$$X(t) = \frac{A}{N}\sum_{i=1}^{N} \cos(\varphi_i(t)), \qquad (5)$$

where $\varphi_i(t)$ denotes the phases of oscillators generated from each node of the network, and $A$ is the signal amplitude, set equal to unity in the model system.

Characteristic (5) can in the first approximation be treated as an analog of the integral signal of an encephalogram characterizing the contribution from a certain group of oscillators (e.g., the neurons in a neuron network) to the signal registered in an experiment from network of oscillators (1). Signal $X(t)$ is analyzed using continuous wavelet transform [14]

$$W(s, \tau) = \int_{-\infty}^{\infty} X(t)\psi^*(s, \tau) dt, \qquad (6)$$

where * denotes complex conjugation,

$$\psi(s, \tau) = \frac{1}{\sqrt{s}}\psi_0\left(\frac{t-\tau}{s}\right) \qquad (7)$$

Equation (7) is a wavelet basis in which $\psi_0$ is the maternal wavelet, $\tau$ is the shift parameter, and $s$ the time scale. In this work, we use the Morlet maternal wavelet, since it is best applied to problems of time-frequency analysis and pattern recognition in signals of a physiological nature [14]:

$$\psi_0(\eta) = \pi^{-\frac{1}{4}} e^{j\omega_0\eta} e^{-\frac{\eta^2}{2}}, \qquad (8)$$

where the central frequency is set equal to $\omega_0 = 2\pi$. In this case, the relation between the frequency of the Fourier transform and the scale parameter (temporal scale $s$) can be written as $f = 1/s$.

## RESULTS FROM ANALYZING PHASE SYNCHRONIZATION USING INTEGRAL CHARACTERISTICS

Let us consider the results from the numerical modeling of adaptive network of coupled Kuramoto oscillators (1). A network of 150 oscillators was studied. Values $\delta_i$ of the Kuramoto oscillators' angular frequencies are set randomly in the range $[0, 2\pi]$. The parameter of oscillator coupling varies within range $0 \le \lambda \le 3.5$ with constant step $\Delta = 0.1$.

We can observe nonsynchronous behavior in the elements of the Kuramoto network on the wavelet surfaces at low values of the coupling parameter ($0 \le \lambda \le 1.5$), but even at $\lambda = 1.5$ we can see three clearly expressed clusters. As was shown in [11, 14], the characteristic rhythms detected from the wavelet spectrum as coupling parameter $\lambda$ grows are associated with the process of clustering. As a result, the mode of global phase synchronization is established upon a further increase in the coupling parameter. The calculated time-aver-

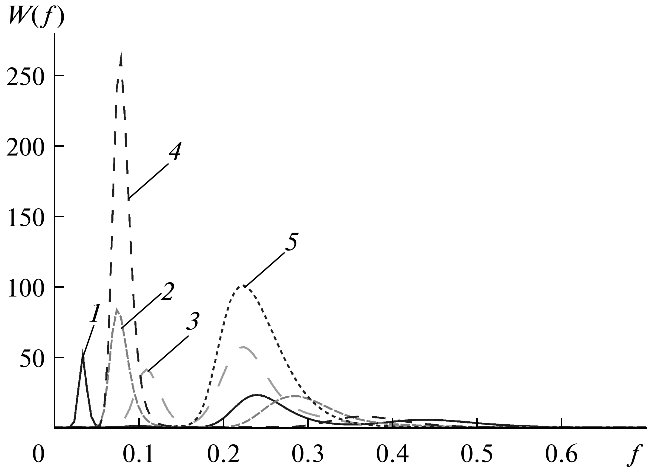


**Fig. 1.** Wavelet spectrum of integral signal (5) for different λ. Curves *1* to *5* correspond to $\lambda = 1.5$, 2, 2.5, 3, and 3.5, respectively.

aged wavelet spectra of the integral signal are shown in Fig. 1 for several values of λ. Curves 1 to 5 correspond to $\lambda = 1.5$, $\lambda = 2$, $\lambda = 2.5$, $\lambda = 3$, and $\lambda = 3.5$, respectively. Each maximum on the wavelet spectrum corresponds to an emerging cluster. We can clearly see that as coupling constant λ grows, three clusters form in the adaptive network. Their number falls as the coupling parameter grows, leading to the mode of global phase synchronization in the network and, as a consequence, to the formation of a single cluster with all elements of the analyzed network.

Figure 2 shows the evolution of the phase distribution of our cluster network. This distribution allows us to determine not only qualitatively but also quantitatively the number of oscillators linked in the clusters of adaptive network. Comparing the wavelet spectra of the integral signal (Fig. 1) and phase distributions of network clusters (Fig. 2), we can see that when $\lambda = 1.5$, the peak with maximum amplitude corresponds to the phase distribution with maximum dispersion, while the second (in height) peak on the wavelet spectrum of the signal corresponds to the phase distribution with minimum dispersion. The investigated clusters are approximately equal in the number of oscillators involved. Where $\lambda = 2.5$ (Fig. 2c), we can see that the clusters also have the same number of oscillators, since we observe equality between the dispersion and amplitude of this characteristic of the dependence. Strong coupling corresponds to a single cluster with a Gaussian phase distribution of oscillators as well (Fig. 2e).

Table 1 shows the results from our study of a Kuramoto network with adaptively changing coupling. The results show the varying amplitude of the wavelet spectrum depending on the strength of coupling, dispersion σ of the phase distribution, and number *N* of oscillators in separate clusters. We can see a clear relation between the number of elements in each cluster

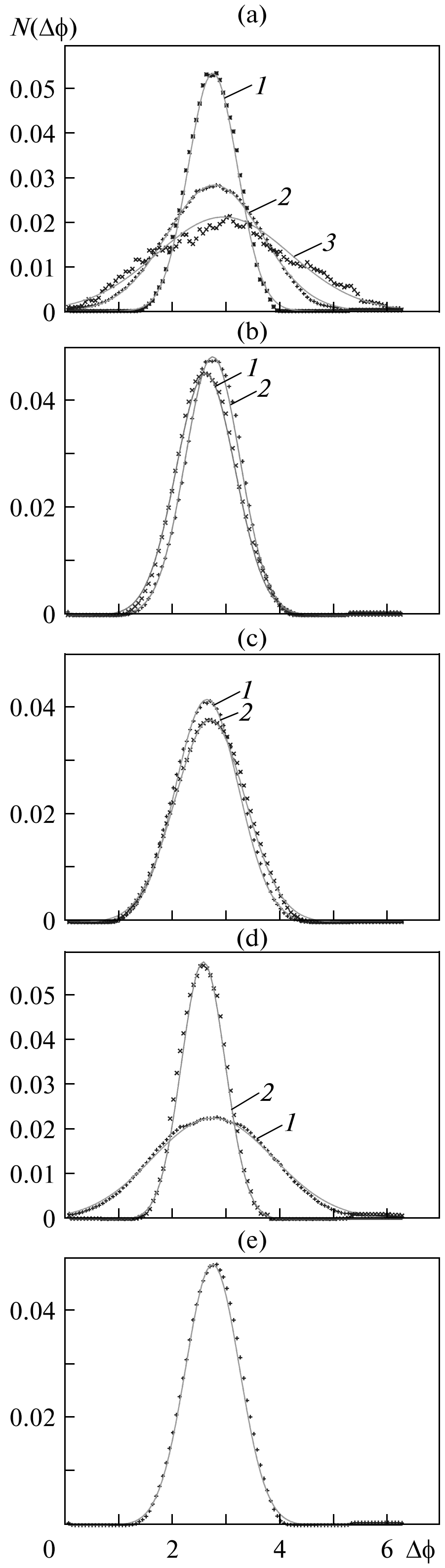


**Fig. 2.** Distribution of the phase difference between the Kuramoto oscillators in an adaptive network for different values of coupling parameter λ: (a) 1.5, (b) 2, (c) 2.5, (d) 3, (e) 3.5. The numbers on the graph refer to the clusters that emerge in the network. The solid curve corresponds to the Gaussian function used in fitting distributions.

Results from studying the clusters in an adaptive network: ε is the coupling strength, $A$ is the peak amplitude on the wavelet spectrum of the signal, σ is the phase dispersion of distribution, and $N$ is the number of oscillators in the cluster

| ε | $A$ | σ | $N$ |
|---|---|---|---|
| 1 cluster | | | |
| 1.5 | 51.3 | 1.3 | 31 |
| 2 | 83.4 | 0.51 | 53 |
| 2.5 | 41.1 | 0.66 | 71 |
| 3 | 261.1 | 1.13 | 115 |
| 2 cluster | | | |
| 1.5 | 23 | 0.46 | 30 |
| 2 | 22.3 | 0.55 | 97 |
| 2.5 | 57 | 0.58 | 79 |
| 3 | 7.8 | 0.41 | 35 |
| 3.5 | 100.6 | 0.5 | 150 |
| 3 cluster | | | |
| 1.5 | 5.5 | 0.89 | 89 |

and the amplitude of the corresponding rhythm in the integral signal. This opens up possibilities for using integral parameters in diagnostics of cluster characteristics, especially the number of elements belonging to a particular cluster with phase synchronization that form in an adaptive network.

## CONCLUSIONS

Phase synchronization in an adaptive network of phase Kuramoto oscillators was analyzed by studying the spectra of the integral wavelet signal and the evolution of the distribution of the phase difference in selected synchronous clusters. It was shown that diagnostics of synchronization using the wavelet spectra and comparing them to the distribution of the phase difference of oscillators yields correct qualitative and quantitative descriptions of the processes of clustering at high values of the coupling parameter in an adaptive network. This approach allows us to determine the number of interacting oscillators in one cluster or another.

The practical significance of these results is associated with using our approach in analyzing real objects that contain large numbers of network elements when the experimental data are limited to using integral characteristics: signals from electro- and magnetoencephalograms, certain types of biological populations, and social and technological networks. In the future, we plan to analyze biological data using the results in this work.

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