ISSN 1063-7850, Technical Physics Letters, 2015, Vol. 41, No. 7, pp. 683–686. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © A.A. Koronovskii, O.I. Moskalenko, A.O. Sel'skii, A.E. Hramov, 2015, published in Pis'ma v Zhurnal Tekhnicheskoi Fiziki, 2015, Vol. 41, No. 14, pp. 39–46.

The Boundary of Generalized Synchronization in Complex Dynamic Systems

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Received February 27, 2015

Abstract—The character of a boundary of the domain of generalized synchronization (GS) regime has been studied for a system of three chaotic oscillators, two which are unidirectionally coupled with the third. It is established that the position of the GS boundary on the plane of coupling parameters is determined by the detuning of frequencies of the interacting chaotic oscillators. The character of this arrangement is explained in the framework of the modified system approach.

DOI: 10.1134/S106378501507024X

The phenomenon of generalized synchronization (GS) in dynamic systems [1, 2] is a partial case of chaotic synchronization [3]. GS is an interesting nonlinear phenomenon that is encountered in many systems [4–6]. In recent years, the interest of researchers expanded from studying chaotic synchronization in separate oscillators to synchronous behavior in complex networks of chaotic oscillators [7–10], including the GS regimes in such systems [11–13]. Evidently, in the presence of many interacting oscillators, the situation becomes more complicated as compared to the case of two coupled oscillators, in particular with respect to the diagnostics and description of this particular type of synchronous behavior (see, e.g., [12] and references therein). At the same time, it can be expected that the main mechanisms responsible for the establishment of GS [2] in a system of two coupled oscillators will also play a determining role in a more complicated structure of interacting oscillators, while this complexity will determine specific features of the appearance of GS regime.

The present work is devoted to the establishment of GS regime and the character of the boundary of these regimes on the plane of control parameters in the particular case in which a driven oscillator is under the action of two master oscillators, which only differ by the values of their control parameters and are not mutually coupled.

Let us consider the dynamics of a system of three chaotic Rössler oscillators (1)–(3), two of which (indicated by subscripts 1 and 2) are independent master oscillators unidirectionally coupled with the third (driven) oscillator (indicated by subscript r):

$$\begin{aligned} \dot{x}_1 &= -\omega_1 y_1 - z_1, \\ \dot{y}_1 &= \omega_1 x_1 + a y_1, \\ \dot{z}_1 &= p + z_1(x_1 - c); \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{x}_2 &= \omega_2 y_2 - z_2, \\ \dot{y}_2 &= \omega_2 x_2 + a y_2, \\ \dot{z}_2 &= p + z_2(x_2 - c); \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{x}_r &= -\omega_r y_r - z_r + \varepsilon_1(x_1 - x_r) + \varepsilon_2(x_2 - x_r), \\ \dot{y}_r &= \omega_r x_r + a y_r, \\ \dot{z}_r &= p + z_r(x_r - c), \end{aligned} \tag{3}$$

where ω_1, ω_2, and $\omega_r = 0.95$ are the parameters determining eigenfrequencies of the corresponding oscillators; $a = 0.15$, $p = 0.2$, and $c = 10$ are the control parameters; and ε_1 and ε_2 are the parameters of coupling between the driving and driven subsystems.

If the eigenfrequencies of driving oscillators coincide ($\omega_1 = \omega_2$), the behavior of system (1) can be reduced (with a certain error) to the well-known case of interaction between two unidirectionally coupled oscillators with coupling parameter $\varepsilon = \varepsilon_1 + \varepsilon_2$. Indeed, according to [2], the mechanism of synchronization between two coupled oscillators can be explained by introducing a modified system. According to this approach, a term describing the dissipative coupling of systems is split into two so as to separately consider the external action of the driving subsystem and the additional dissipation that plays a key role in establishment of the GS regime. Indeed, the GS appears when an increase in the coupling parameter introduces additional dissipation into the system, which is sufficient to suppress intrinsic chaotic dynamics in the modified system (see [2]). By analogy,

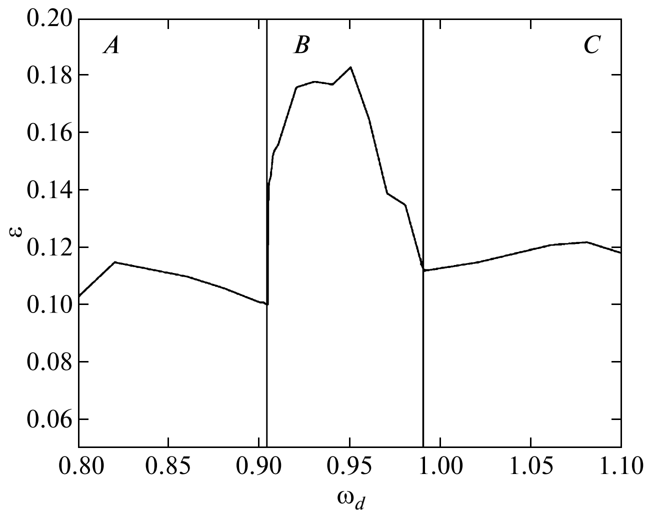

Fig. 1. GS boundary for two unidirectionally coupled Rössler oscillators [14], which reveals regions *A*, *B*, and *C* with different characteristic values of the coupling parameter necessary for the establishment of a GS regime (see text for explanations).

it is possible to introduce a modified system in the case of three interacting oscillators under consideration. In the case of identical frequencies of the driving oscillators ($\omega_1 = \omega_2$), an increase in coupling parameter ε to certain critical value ε_c leads to the passage of the modified system to a periodic regime and the establishment of GS regime in the driven subsystem. Accordingly, the boundary of GS regimes on the plane of parameters ($\varepsilon_1, \varepsilon_2$) for system (1) is determined by the relation $\varepsilon_2 = \varepsilon_c - \varepsilon_1$ and represented by a straight line.

However, if the parameters (and eigenfrequencies) of driving oscillators (1) and (2) do not coincide, the situation becomes different because the appearance and breakage of GS is determined not only by the additional dissipation related to coupling, but also by the external action that can (in contrast to dissipation) excite the intrinsic chaotic dynamics in the modified system, thus breaking the synchronous regime. It is the external action that accounts for a specific character of the boundary of GS regimes in the case of interaction between two unidirectionally coupled chaotic oscillators [12]. Since this boundary reflects the influence of a driving action on the excitation of intrinsic chaotic dynamics in the driven subsystem, this factor plays an important role in the case of three interacting oscillators under consideration.

Figure 1 shows the GS boundary for two unidirectionally coupled Rössler oscillators (see also [14, 15]). Calculations were performed using the auxiliary system method [16] with additional refinement of Lyapunov exponents of the systems under consideration. As can be seen, the plane of control parameters (ω_d, ε) contains several characteristic regions. In region *B*, the establishment of GS regime requires a relatively large value of the coupling parameter. In regions *A* and *C*, the GS regime is established at significantly lower ε values. A significant difference of ε_c values corresponding to the GS boundary in region *B* is related to the excitation of chaotic oscillations in the driven subsystem [14].

In studying the influence of two driving Rössler oscillators on the third (driven) oscillator, let us consider various cases of frequency detuning between these subsystems:

(i) parameters ω_i of both driving oscillators occur in regions *A* or *C*;

(ii) parameter ω_i of one driving oscillator falls in region *B*, while ω_i of another driving oscillator occurs in region *A* or *C*; and

(iii) both ω_i values of driving oscillators belong to region *B*.

Figure 2a shows the boundaries of GS regimes in the first case. As can be seen, the GS boundaries on the plane of coupling parameters (ε_1, ε_2) are close to the diagonal. This is especially clear for curve *4*, which corresponds to close values of ω_1 and ω_2 (and the eigenfrequencies of two driving oscillators) and, hence, close values of the coupling parameters necessary for the establishment of GS. However, it should be emphasized that the GS boundary is also close to the diagonal when ω_1 and ω_2 of the driving oscillators belong to different regions (*A* and *C*). This situation is close to the case of two identical driving oscillators: since each separate oscillator has almost the same critical value of the coupling parameter for the appearance of GS, these oscillators are also equal in respect of both introducing additional dissipation and exciting intrinsic chaotic dynamics in the driven subsystem.

A different behavior is observed when parameter ω_i of one of the two driving oscillators belongs to region *B*, while ω_i of another driving oscillator falls in region *A* or *C*. In this case (Fig. 2b), the GS boundary is far from the diagonal. On the whole, these combinations of frequencies are characterized by “bending” of the GS boundary on the (ε_1, ε_2) plane, which can be subdivided into two approximately linear regions with different slopes. Comparison of Figs. 2a and 2b leads to the conclusion that the character of GS boundary in this case depends not only on the difference of frequencies of the driving oscillators, but also on the difference of their coupling parameters in Fig. 1. The bending of the GS boundary is explained by different properties of the driving oscillators, which lead to the excitation of intrinsic chaotic oscillations in the driven subsystem. For nearly the same coupling parameters ε_i, both driving oscillators introduce approximately equal additional dissipation in the driven subsystem, but the driving oscillator with ω_i in region *B* will excite the intrinsic chaotic dynamics to a greater extent.

In the third important case (Fig. 2c), the ω_i values for both driving oscillators belong to region *B*. Pro-

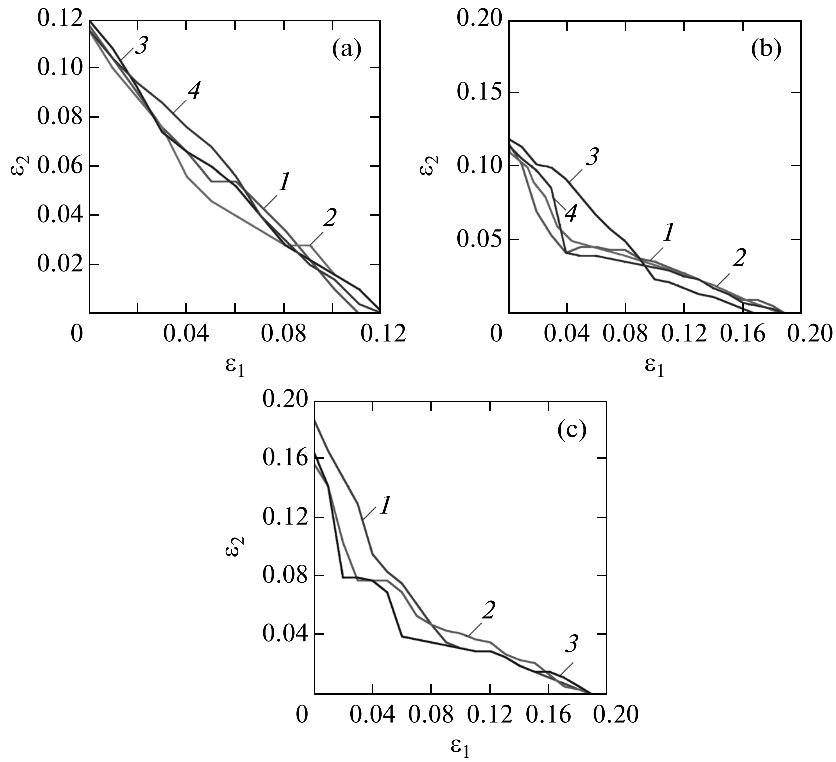

Fig. 2. GS boundaries in the system of three Rössler oscillators, plotted on the plane of coupling parameters (ε_1, ε_2) for various frequencies of driving subsystems:
(a) $\omega_1 = 1.00$, $\omega_2 = 0.82$ (*1*); $\omega_1 = 1.05$, $\omega_2 = 1.02$ (*2*); $\omega_1 = 0.82$, $\omega_2 = 1.05$ (*3*); $\omega_1 = 1.00$, $\omega_2 = 1.02$ (*4*); (b) $\omega_1 = 0.92$, $\omega_2 = 0.82$ (*1*); $\omega_1 = 0.93$, $\omega_2 = 1.05$ (*2*); $\omega_1 = 0.96$, $\omega_2 = 1.00$ (*3*); $\omega_1 = 0.92$, $\omega_2 = 1.02$ (*4*); (c) $\omega_1 = 0.92$, $\omega_2 = 0.93$ (*1*); $\omega_1 = 0.93$, $\omega_2 = 0.91$ (*2*); $\omega_1 = 0.92$, $\omega_2 = 0.96$ (*3*).

ceeding from the modified system approach and taking into account the above considerations, it would be logical to expect that the GS boundary in this case must also be close to the diagonal (as in Fig. 2a), but the observed pattern is different. It should be noted that the GS boundary exhibits bending and one can even distinguish three approximately linear segments rather than two segments as in Fig. 2b. This behavior is again explained by different values of ε_i at which intrinsic chaotic dynamics is excited in the driven subsystem (for more detail, see [14]) for various ω_i values, even although these values are close and belong to the same region (*B*).

On the whole, it is concluded that, in the case of two identical unidirectionally coupled Rössler oscillators with different eigenfrequencies acting on the third (driven) oscillator, the GS boundary on the plane of coupling parameters can be far from the diagonal observed in the case of coinciding frequencies of the two driving oscillators. In the case in which control parameters of both driving oscillators occur in regions *A* or *C*, the GS boundary is actually close to the diagonal. When the parameters of both systems belong to region *B*, the situation changes and the boundary acquires a rather complicated character even despite quite close frequencies. In the case when parameter ω_i of one of the two driving oscillators belongs to region *B*, while ω_i of another driving oscillator falls in region *A* or *C*, the GS boundary can be subdivided into two approximately linear regions with different slopes depending on the particular frequencies. In this case, there are no significant differences between oscillators with the parameters in regions *A* and *C*.

Acknowledgments. This study was supported in part by the Russian Science Foundation, project no. 14-12-00224. The work of A. Sel'skii was supported by the nonprofit Dynastiya foundation, and A. Hramov gratefully acknowledges support from the Ministry of Education and Science of the Russian Federation in the framework of project no. 931.

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Translated by P. Pozdeev