ISSN 1063-7850, Technical Physics Letters, 2015, Vol. 41, No. 8, pp. 765–767. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © A.A. Koronovskii, O.I. Moskalenko, A.A. Pivovarov, A.E. Hramov, 2015, published in Pis'ma v Zhurnal Tekhnicheskoi Fiziki, 2015, Vol. 41, No. 16, pp. 1–7.

Establishment of Generalized Synchronization in a Network of Logistic Maps

A. A. Koronovskii[a, b]*, O. I. Moskalenko[a, b], A. A. Pivovarov[a], and A. E. Hramov[a, b]
[a] *Saratov State University, Saratov, 410012 Russia*
[b] *Yury Gagarin State Technical University of Saratov, Saratov, 410054 Russia*
**e-mail: alexey.koronovskii@gmail.com*
Received March 17, 2015

Abstract—We have studied the process of generalized chaotic synchronization establishment in a network of mutually coupled logistic maps and analyzed the character of interaction between elements of the network on the passage from asynchronous to synchronous dynamics related to increase in the coupling parameter. Peculiarities of the interaction between elements of the network and the onset of generalized synchronization have been elucidated using the method of phase tubes.

DOI: 10.1134/S1063785015080246

Among the various types of chaotic synchronization [1], such as the phase synchronization, lag synchronization, and total synchronization, the phenomenon of generalized synchronization (GS) in dynamical systems has drawn considerable interest of researchers [2–4]. GS can arise between various interacting oscillators, including those with different phase spaces dimensions. A criterion for the establishment of a GS regime is the existence of a functional relationship between states of the interacting oscillators [2, 5]. At present, the GS phenomenon has been studied in sufficient detail for a large class of interacting systems, including the unidirectionally and mutually coupled systems with discrete time [5], and the unidirectionally [6] and mutually [7] coupled (in particular, spatially distributed) flow systems [8–10].

The next step in investigations of the GS phenomenon was related to the passage from separate oscillators to networks of nonlinear elements. However, the specific features and complexity of such systems [7] the situation becomes more complicated and poses numerous general questions. One of these, related to the character of GS establishment in networks of nonlinear elements, has been studied in the present work.

Let us consider the dynamics of a simple model network of coupled nonlinear elements, namely, five ($x_1, x_2, ..., x_5$) mutually coupled logistic maps:

$$x_{in+1} = f(x_{in}, a_i) + \varepsilon C_{ij} \sum (f(x_{jn}, a_j) - f(x_{in}, a_j)), \quad (1)$$

where $f(x, a) = ax(1-x)$; $i, j = 1, ..., 5$ are indices enumerating the network elements; C_{ij} are elements of the coupling matrix ($C_{ij} = 0$ implies the absence of coupling, $C_{ij} = 1$ corresponds to the interaction of each element with all others); and ε is the coefficient of coupling. The value of coefficient ε is the same for all elements and serves a control parameter that determines a regime that is established in the system, while the other control parameters a_i ($i = 1, ..., 5$) have been selected differently: $a_1 = 3.75$, $a_2 = 3.76$, $a_3 = 3.77$, $a_4 = 3.78$, and $a_5 = 3.79$. The topology of network coupling was set as corresponding to the mutual ("each one with all others") coupling: $C_{ij} = 1$ (Fig. 1).

A classical approach to detecting the GS regime in a system of unidirectionally coupled oscillators is based on the nearest neighbor method [5]. However, for the reasons considered in [7, 11, 12], this method

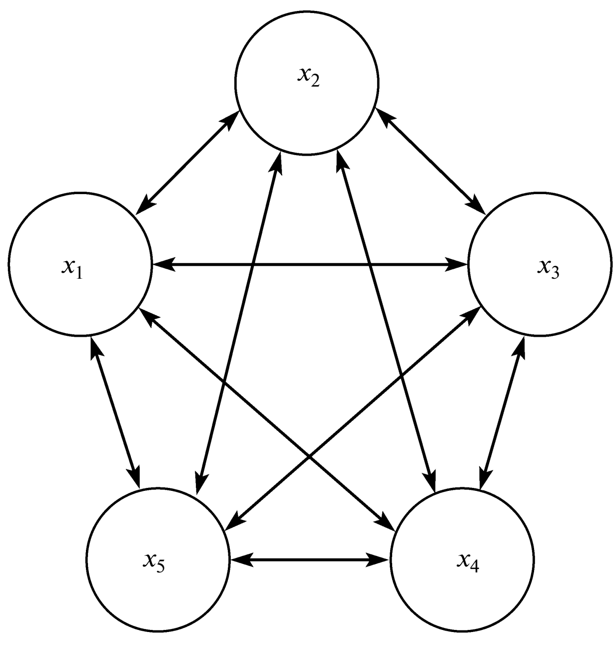

Fig. 1. Topology of the network of five logistic maps under consideration.

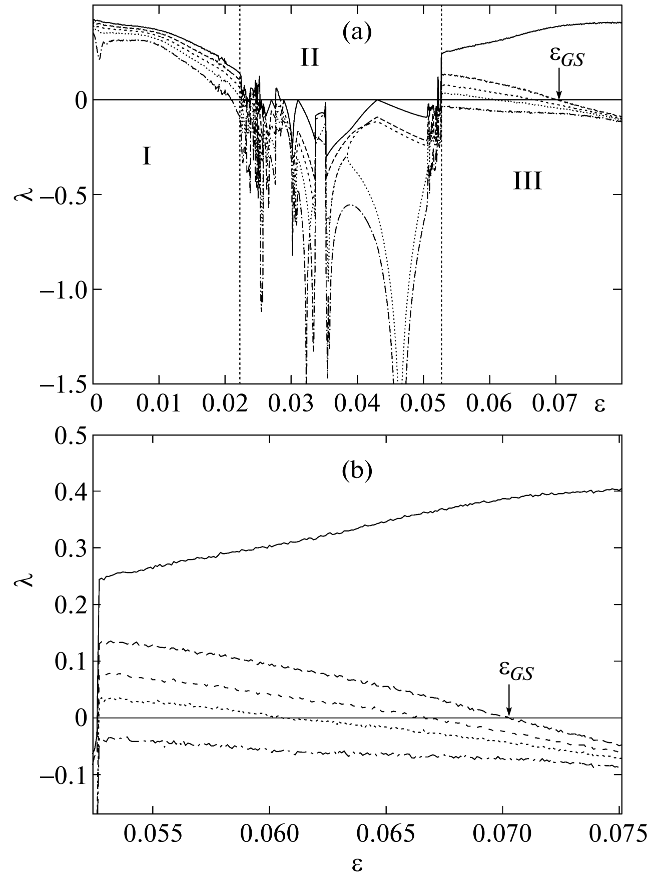

Fig. 2. (a) Dependence of five Lyapunov exponents on the coupling coefficient ε in the system of mutually coupled logistic maps and (b) region III plotted on a greater scale. Key: GS (roman subscript)

turns out to be inapplicable to studying the interaction between mutually coupled systems. Therefore, it is necessary to use other methods, for example, calculation of the Lyapunov exponents [11, 13]. According to this, a criterion of GS in the case of a pair of coupled oscillators is that one of the two largest Lyapunov exponents passes via zero to the region of negative values, while the other Lyapunov exponent remains positive [6, 14] (a situation with two positive Lyapunov exponents for autonomous oscillators, known as the regime of hyperchaotic oscillations, is not considered here). In the case of a network comprising N nonlinear elements at small values of the coupling parameter, the spectrum of Lyapunov exponents consists of N positive quantities. As the coupling parameter increases, these values sequentially pass to the negative region [7]. In particular, for the network of five mutually coupled logistic maps, a criterion for the onset of GS regime in this system is the passage of four Lyapunov exponents to the region of negative values while the first exponent remains positive.

Figure 2a shows the dependence of five Lyapunov exponents on coupling coefficient ε in the system of five mutually coupled logistic maps (1). As can be seen, the plot of $\lambda(\varepsilon)$ can be divided into three regions (Fig. 2a): (I) $\varepsilon \in [0; 0.023]$, region of asynchronous dynamics; (II) $\varepsilon \in [0.023; 0.054]$, periodicity windows; and (III) $\varepsilon \in [0.054; 0.071]$, region of transition to a GS regime. During this transition, four Lyapunov exponents sequentially change sign and cross the abscissa axis (Fig. 2b). The critical moment of transition to the GS regime (change of sign of the second largest Lyapunov exponent) at the coupling parameter of $\varepsilon_{\mathrm{GS}} \approx 0.0703$ is indicated by the arrow in Fig. 2b. Thus, at the boundaries of region III, the system shows asynchronous ($\varepsilon < 0.054$) and synchronous ($\varepsilon > 0.071$) dynamics. At the same time, in contrast to the case of two coupled oscillators, a sharp threshold between the asynchronous and synchronous behavior is absent and the dynamics of network elements exhibits gradual transition, the main mechanisms and laws of which are still unknown.

Processes taking place in region III, where some of the Lyapunov exponents have already changed sign, can be elucidated using the method of phase tubes [5, 12]. The idea of this method consists in considering the dynamics of coupled oscillators by selecting a reference point x_n^l and tracing its prehistory over K preceding steps x_{n-k}^l ($k = 0, ..., K$). Then, nearest neighbor x_j^l of the reference point is found for which the phase trajectories x_{n-k}^l and x_{j-k}^l ($k = 0, ..., K$) are close in the entire prehistory interval K, i.e., the two phase trajectories occur inside a phase tube of length K [5, 12]. Provided that the condition of proximity of the phase trajectories is obeyed, nearest neighbors x_j^m in phase spaces of the rest of the oscillators are fixed and the numerical characteristic of synchronism is determined as the variance of distances between nearest neighbors x_j^m.

For this investigation, two values of the coupling parameter were selected as most interesting: $\varepsilon_1 = 0.058$ and $\varepsilon_2 = 0.068$. For $\varepsilon = \varepsilon_1$, only one of the five Lyapunov exponents is negative, whereas at ε_2 three exponents have passed to the negative region (Fig. 2b). According to the method of phase tubes described above, all nodes of the network under consideration were studied in pairs so as to obtain a clear pattern of the character of their interaction. The values of variance of the distribution of distances between nearest neighbors in various nodes for the two selected values of ε are presented in Tables 1 and 2.

The column number in Tables 1 and 2 corresponds to the node in which reference point x_n^l is selected, while the row numbers indicate nodes for which the character of interaction is determined. Analysis of these data reveals that the values of variance for some pairs of network nodes are sharply different from other

Table 1. Values of variance for coupling parameter $\varepsilon_1 = 0.058$

	x_1	x_2	x_3	x_4	x_5
x_1	–	3.021×10^{-7}	3.788×10^{-7}	2.005×10^{-7}	6.002×10^{-7}
x_2	1.232×10^{-6}	–	4.622×10^{-6}	1.103×10^{-6}	1.061×10^{-6}
x_3	2.402×10^{-6}	2.310×10^{-6}	–	2.190×10^{-6}	3.133×10^{-6}
x_4	5.664×10^{-6}	3.724×10^{-6}	1.232×10^{-6}	–	7.023×10^{-6}
x_5	2.891×10^{-6}	5.152×10^{-6}	7.617×10^{-6}	3.686×10^{-6}	–

Table 2. Values of variance for coupling parameter $\varepsilon_2 = 0.068$

	x_1	x_2	x_3	x_4	x_5
x_1	–	1.326×10^{-7}	1.533×10^{-7}	2.513×10^{-7}	2.763×10^{-7}
x_2	1.099×10^{-7}	–	1.592×10^{-7}	2.990×10^{-7}	3.102×10^{-7}
x_3	2.264×10^{-7}	1.278×10^{-7}	–	4.908×10^{-7}	3.565×10^{-7}
x_4	8.002×10^{-7}	7.393×10^{-7}	6.773×10^{-7}	–	5.745×10^{-7}
x_5	3.348×10^{-6}	3.960×10^{-6}	3.496×10^{-6}	3.841×10^{-6}	–

values. In accordance with the nearest neighbor method, it can be concluded that a small variance (in this case, $\sim 10^{-7}$) is indicative of the synchronization of two elements in the GS sense, whereas a large variance ($\sim 10^{-6}$) indicates that the two elements are not yet synchronized. In the given case, the conclusion is that, for $\varepsilon_1 = 0.058$, four elements of the five demonstrate synchronism with the first element, while this first is not synchronized with the others. The reverse situation is observed for $\varepsilon_2 = 0.068$, where the fifth element demonstrates asynchronous behavior, while the other elements are synchronized with each other.

Thus, we have studied the phenomenon of GS establishment in a network of mutually coupled logistic maps. The character of interaction between nonlinear elements of the network in the region of transition from asynchronous dynamics to the GS regime was elucidated using the calculation of Lyapunov exponents and the method of phase tubes,

Acknowledgments. This study was supported in part by the Russian Science Foundation, project no. 14-12-00224. The work of A. Hramov was also supported by the Ministry of Education and Science of the Russian Federation in the framework of project no. 931.

REFERENCES

1. A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
2. N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
3. B. S. Dmitriev, A. E. Hramov, et al., Phys. Rev. Lett. **102**, 074101 (2009).
4. R. A. Filatov, A. E. Hramov, and A. A. Koronovskii, Phys. Lett. A **358**, 301 (2006).
5. A. A. Koronovskii, O. I. Moskalenko, S. A. Shurygina, and A. E. Hramov, Chaos Soliton. Fract. **46**, 12 (2013).
6. H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, Phys. Rev. E: **53**, 4528 (1996).
7. O. I. Moskalenko, A. A. Koronovskii, A. E. Hramov, and S. Boccaletti, Phys. Rev. E **86**, 036216 (2012).
8. A. A. Koronovskii, P. V. Popov, and A. E. Khramov, J. Exp. Theor. Phys. **103** (4), 654 (2006).
9. O. I. Moskalenko, A. E. Hramov, A. A. Koronovskii, and A. A. Ovchinnikov, Europhys. J. B **82** (1), 69 (2011).
10. A. E. Hramov, A. A. Koronovskii, and P. V. Popov, Phys. Rev. E **72** (3), 037201 (2005).
11. O. I. Moskalenko, A. A. Koronovskii, and A. E. Hramov, Phys. Rev. E **87**, 064901 (2013).
12. A. A. Koronovskii, O. I. Moskalenko, and A. E. Hramov, Phys. Rev. E **84** (3), 037201 (2011).
13. S. P. Kuznetsov, *Dynamical Chaos* (Fizmatlit, Moscow, 2001) [in Russian].
14. K. Pyragas, Phys. Rev. E **56**, 5183 (1997).

Translated by P. Pozdeev

SPELL OK