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Digital Filtering of Noisy Data Concerned with the Effect of the Threshold Level and Choice of a Wavelet

A. S. Yasin[a, b], A. N. Pavlov[a, c, d], and A. E. Hramov[a, c, d]

[a]Chernyshevskii State University, ul. Astrakhanskaya 83, Saratov, 410012 Russia
[b]University of Technology, Al-Sina'a street, Baghdad, Republic of Iraq
[c]Yuri Gagarin Saratov State Technical University, ul. Politekhnicheskaya 77, Saratov, 410054 Russia
[d]Kotel'nikov Institute of Radio Engineering and Electronics (Saratov Branch), Russian Academy of Sciences, ul. Zelenaya 38, Saratov, 410019 Russia
e-mail: pavlov_lesha@mail.ru

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Abstract—The problem of digital wavelet filtering of noisy data is considered. The results are comparatively analyzed when the threshold function is specified in different ways for wavelet transformation coefficients. The influence of the choice of the wavelet basis on the efficiency of the noise removal is studied. It is shown that the minimum filtering error is in most cases provided when the soft variant of the threshold function introduction is introduced.

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INTRODUCTION

Digital filtering of noisy signals and images is important for a wide circle of scientific and technical problems. These problems arise, in particular, in the communication technique for improving the quality of the reception of transmitted reports. Recently, they are more often solved with the use of approaches based on discrete wavelet transformation (DVT) [1–3]. These approaches enable us to efficiently remove localized noises whose filtering with the application of the Fourier transformation is not efficient because the basis of infinitely oscillating functions is used. Therefore, the aforementioned approaches have advantages over filters based on the Fourier transformation. The DVT widely applied in the multiple-scale analysis makes it possible to decompose a signal or an image into components corresponding to different observation scales. After that, the decomposition coefficients corresponding to small scales that are mostly subjected to the noise influence can be corrected [4]. In the simplest case, the corresponding coefficients are put equal to zero by analogy with the registration of graphical data in the JPEG2000 format, which also uses the decomposition in wavelets and rejection of the least informative wavelet coefficients if it is necessary to contract an image with an insignificant quality loss. Thus, the effect of the image compression and filtering of noises that are present in small scales is simultaneously reached.

Note that the simple version of zeroing certain coefficients is not always efficient. Therefore, by now, approaches applying the variants of hard and soft introduction of the threshold function in digital filtering have been suggested [4–7]. In the first case, the wavelet coefficients that do not exceed a given threshold level are put to zero, which leads to discontinuities of the threshold function and disturbs the regularity of the restored signal. In the second case, the threshold function has no discontinuities and the values are corrected for all wavelet coefficients. In addition, various modernizations of the methods of wavelet decomposition can be applied. Among these, we can emphasize the method of dual-tree complex wavelet-transform [8, 9]. This approach extends the classic DVT that uses real basis functions such as the Daubechies wavelets. It is an analogy to the analytical signal method, which provides for adding a real signal by an imaginary part.

Despite the development of digital filtering methods that use the wavelet transform, there are a lot of unsolved questions in the practical application of these methods and the choice of a concrete filtering method remains to be a complex problem as before. Therefore, the comparative analysis of various filtering techniques remains to be actual when they are oriented to choosing an approach that minimizes the distortions introduced when the signal or image is restored from its wavelet coefficients. In this study, we discuss how the choice of the wavelet basis and the threshold function in the space of wavelet coefficients affect the quality of digital filtering. With this purpose, we perform a series of test investigations specifying the addition of noise into an image and the subsequent filtering of the latter. This approach makes it possible to find the error of digital filtering of the introduced noise and solve the problem of minimization of this error using the tuning of algorithm parameters.

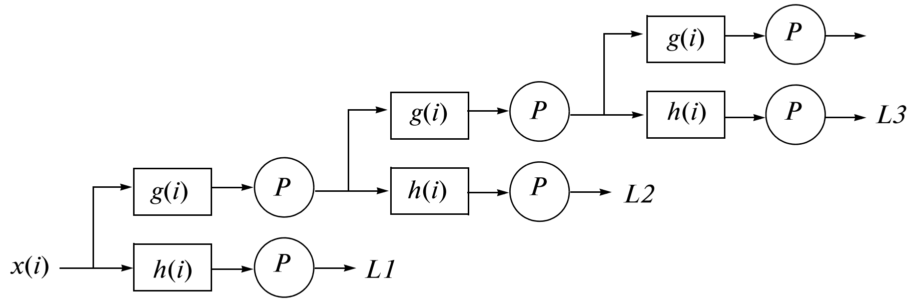

Fig. 1. Procedure of the DVT performed with the help of a set of quadrature mirror filters; P is the procedure of thinning twice a temporal series, L is the resolution level.

1. NOISE FILTERING WITH THE USE OF THE DVT

Mathematical apparatus of discrete wavelet transform is widely used in technique including the transmission and coding of information when quick calculations are necessary. This apparatus differs from the continuous wavelet transform (CWT) [10–14], which is often used for analyzing the structure of signals of systems with characteristics that change in time [15–20]. First, the DVT usually operates with the orthonormal basis that provides for a more compact representation of a signal. This circumstance provides for a less number of decomposition coefficients as compared with the CVT. In addition, the DVT provides for a faster scheme of computations based on the pyramidal algorithm of the signal decomposition. Second, the application of discrete wavelets is not reduced to a usual discretization of the CVT formulas. These two variants of the wavelet transformation apply different bases. If, in the case of the CVT, the basis is constructed with the use of functions having an analytical recording form (for example, derivatives of the Gaussian functions), then, the functions having no analytical continuation are applied within the framework of the DVT. These functions are specified with the use of a set of numbers (filter coefficients), which means the necessity of using matrices. In this work, we apply the family of Daubechies wavelets [1] as basis functions. Let us give an example of the set of the filter coefficients with which Daubechies wavelet D^8 widely applied in practice is specified

$$\begin{gathered} h_0 = -0.0757657, \quad h_1 = -0.0296355, \\ h_2 = 0.4976187, \quad h_3 = 0.8037388, \\ h_4 = 0.2978578, \quad h_5 = -0.0992195, \\ h_6 = -0.0126040, \quad h_7 = 0.0322231. \end{gathered} \tag{1}$$

During realization of the DVT, a signal is supplied at the input of two conjugate quadrature mirror filters. The decomposition according to the pyramidal algorithm is possible, when the volume of a sample is specified in the form of a degree of the number 2, i.e., $N = 2^k$. When the temporal series $x(i) = x(i\Delta t)$ arrives at the input of the low-frequency filter having frequency characteristic $g(i)$, the filtered signal

$$y(i) = \sum_{l=-\infty}^{\infty} x(l)g(i-l) \tag{2}$$

is obtained at the filter output.

A conjugate high-frequency filter with frequency characteristic $h(i)$ is coupled with the low-frequency filter as follows:

$$g(i) = (-1)^i h(2M - i - 1), \tag{3}$$

where M is the length of the wavelet specification region. The filtered signals are then thinned so that only odd or even samples remain, which corresponds to the subband coding scheme [2, 12]. As a result, we obtain two sample sequences

$$y_{\mathrm{LF}}(i) = \sum_{l=-\infty}^{\infty} x(l)g(2i-l), \quad y_{\mathrm{HF}}(i) = \sum_{l=-\infty}^{\infty} x(l)h(2i-l). \tag{4}$$

The thinning is possible, because, when mirror filters are applied, the signal frequency range becomes twice as narrow. The thinned signals again arrive at the filters' input, which results in the successive double reduction of the frequency bandwidth. The DVT procedure is schematically shown in Fig. 1.

As a result of thinning, each of the temporal series is characterized by the frequency range, which is half of the range of the signal before filtering. However, the presence of two sequences (at the output of each filter) makes it possible to uniquely restore the original signal during the inverse transformation.

The application of low-frequency filters can be interpreted as the approximation of a signal at different levels of resolution, and the application of high-frequency filters can be interpreted as detailing with respect to the chosen resolution level. Coefficients $d_{j,k}$ of decomposition in wavelets show the amplitude characteristics of analyzed processes at different resolution levels. To filter noise the small-scale wavelet coefficients that have no high absolute values (and that are mostly open to the fluctuation effect) are thrown

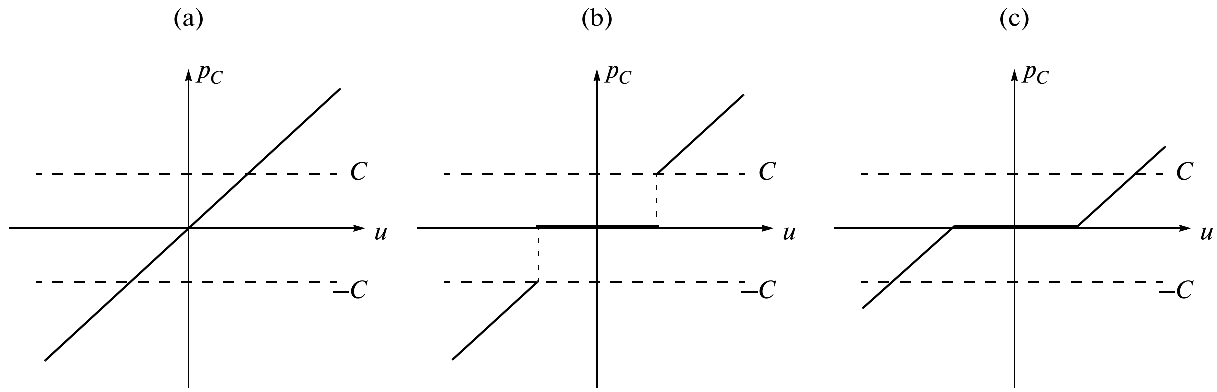

Fig. 2. Specification of threshold function $p_C(u)$: (a) the initial signal, (b) the hard variant of the specification of the threshold function, and (c) the soft variant of the specification of the threshold function.

away before the inverse transformation (the threshold filtering method). Then, the quality of filtering substantially depends on the choice of the specification variant of the threshold function [5–7], which is multiplied by the corresponding coefficients before the inverse transformation (soft or hard, Fig. 2), and on the wavelet basis. The appropriate choice facilitates obtaining a higher quality of the signal or image refinement of noise.

Figures 2a–2c show three variants of specification of threshold function $p_C(u)$ for the wavelet transform coefficients. In variant (a), the equality $p_C(u) = u$ is applied. It means the absence of coefficient corrections. The initial signal is obtained as a result of the inverse transformation. In variant (b), the function is specified in the form

$$p_C(u) = \begin{cases} u, & |u| \geq C, \\ 0, & |u| < C. \end{cases} \tag{5}$$

When this threshold function is applied, the wavelet coefficients that have large modules (i.e., that are the most significant ones) remain invariable and the small wavelet coefficients are zero filled. At last, for variant (c), the threshold function is chosen in the following way:

$$p_C(u) = \begin{cases} u - C, & u \geq C, \\ u + C, & u \leq -C, \\ 0, & |u| \leq C. \end{cases} \tag{6}$$

Note that, in the last case, the decrease of the absolute values of all wavelet coefficients, including those that have high modules, can change the amplitude of the restored signal. This approach cannot be applied to the applications where it is necessary to retain invariable amplitude characteristics. However, there are problems where it is more important to keep the signal regularity than to accurately reproduce its amplitude. A characteristic example is image filtrating of different interferences, where the method of soft specification of a threshold function is a widely used approach.

The above procedure of decomposition in wavelets is more complex in the analysis of images than in the analysis of 1D signals (temporal series). In this case, the image is analyzed on a 2D plane by horizontals, verticals, and diagonals with an identical resolution, and the corresponding filters are formed on the basis of the products of frequency characteristics of low- and high-frequency filters for the 1D case. This procedure is described in more detail, for example, in the review [21]. Upon decomposition in wavelets, the interferences are filtered by the multiplication of the coefficients by one of the threshold functions (see Fig. 2) by analogy with the case of 1D signals.

2. COMPARATIVE ANALYSIS OF THE RESULTS OF WAVELET FILTERING OF NOISY IMAGES

To illustrate the efficiency of the wavelet filtering method, the following investigations are performed. A normal random process (white noise) with the various dispersions is added to the chosen image. Then, filtering of introduced interferences is realized when various wavelet bases and methods of introducing the threshold level are specified and the threshold value is varied. The filtered image is compared to the original image by calculating the square root of the rms error (R) and the search for the minimum error (the best noise suppression) is performed.

At the first stage, we perform the investigation of the effect of the wavelet basis choice. We consider the black-and-white image having the 560×800 dimension (Fig. 3a) and five values of the intensity of the white noise (with the intensity 0.1, 0.2, 0.3, 0.4, and 0.5). At each value of the intensity, we perform the direct DVT in the basis of the Daubechies wavelets

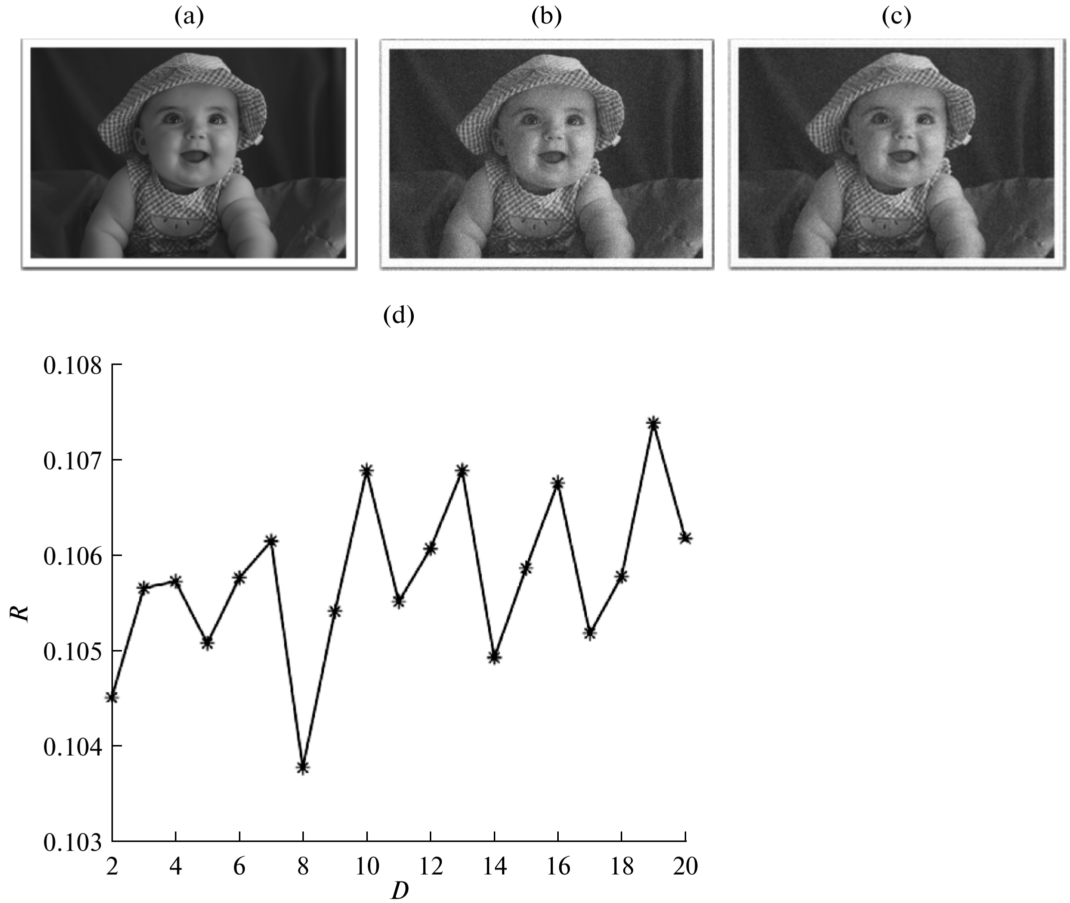

Fig. 3. Images that are (a) initial, (b) noisy, and (c) filtered with the help of wavelet D^8 and (d) the dependence of error R on the choice of the wavelet basis.

(from D^2 to D^{20}). Analogous additional calculations are performed when the image dimension is varied (i.e., when the scaling is changed with different coefficients). The examples of the noisy (the noise dispersion is 0.1) image filtered with wavelet (1) are given in Figs. 3b and 3c. The distinctive dependence of error R on the chosen wavelet basis is shown in Fig. 3d. In this example, wavelet D^8 provides for the minimum mean-square error.

To test the correctness of the above conclusions, analogous calculations are performed for another image (Fig. 4). Despite the fact that the minimum error for wavelet D^8 in Fig. 4d is less expressed as compared with Fig. 3d, we can say that, in this case, also the choice of the corresponding basis should be considered as the optimal one.

Below, analogous calculations are performed for different dimensions of images (the scaling change with coefficients 0.5–1.5). The dimension change yields that the minimum error is reached for different basis functions. However, in many cases, the corresponding minimum is obtained for wavelet D^8, which can be considered as the trade-off between the length of the specification region and regularity of the basis function. Therefore, just this wavelet is used in the calculations below.

In the second stage, we study the influence of the specification of the threshold function and threshold quantity. Figure 5 shows the dependence of error R on threshold level C for hard and soft variants of the threshold function assignment when the dispersion of the white noise added to the image shown in Fig. 4a is 0.1.

It follows from Fig. 5 that the minimum error is reached for the soft variant of the assignment of function $p_C(u)$. This variant provides for the error decrease, which is most expressed for small values of level C. When the intensity of interference added to the image is increased, the differences between the two variants of the threshold function specification become less expressed (see table). Nevertheless, the soft variant of the specification of function $p_C(u)$ provides for the minimum error for all intensities of interference. Similar conclusions are made for various images and during similar calculations performed when the image dimensions are varied (i.e., when the scaling change with coefficients 0.5–1.5 is performed).

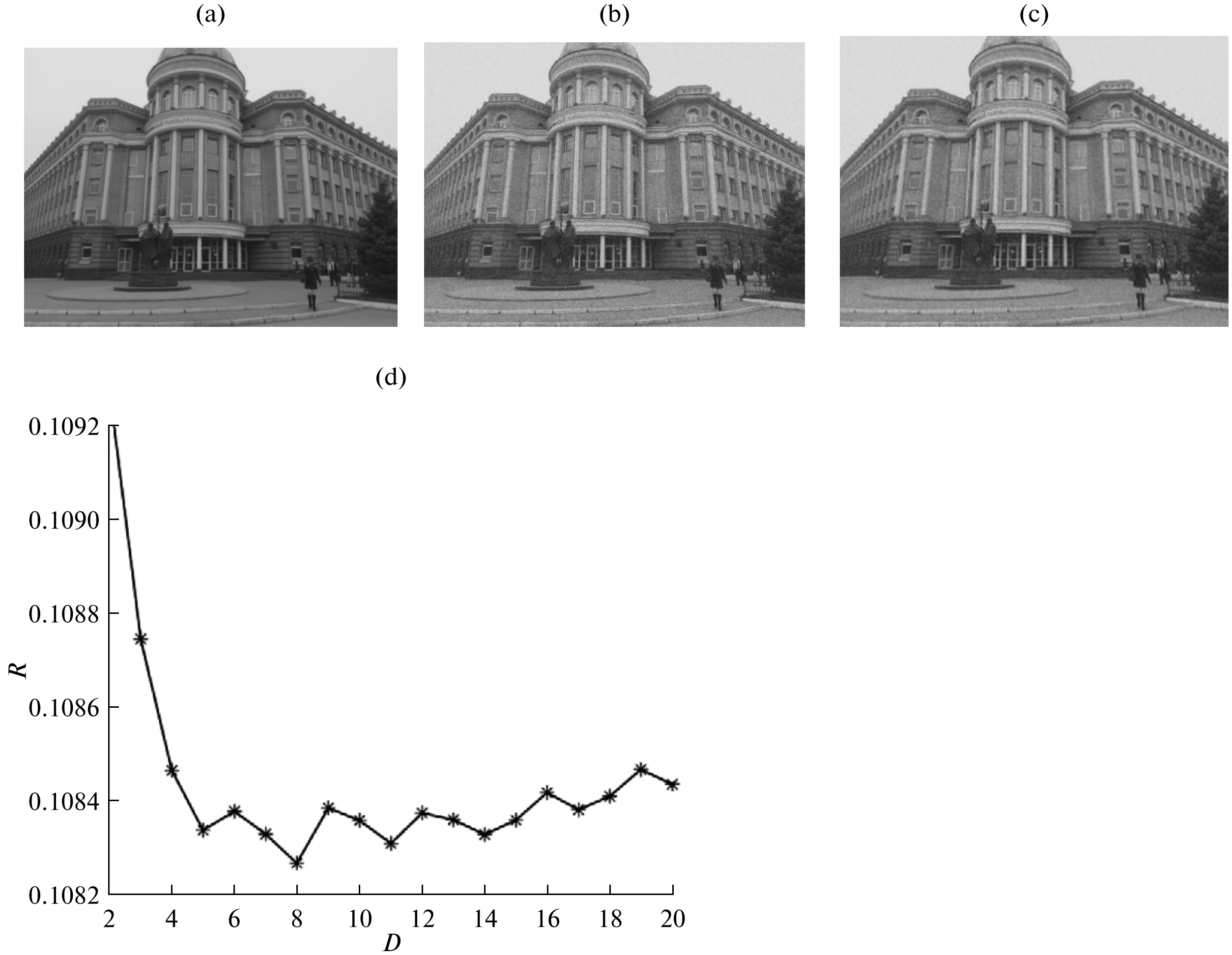

Fig. 4. Images that are (a) initial, (b) noisy, and (c) filtered with the help of wavelet D^8 image and (d) the dependence of error R on the choice of the wavelet basis.

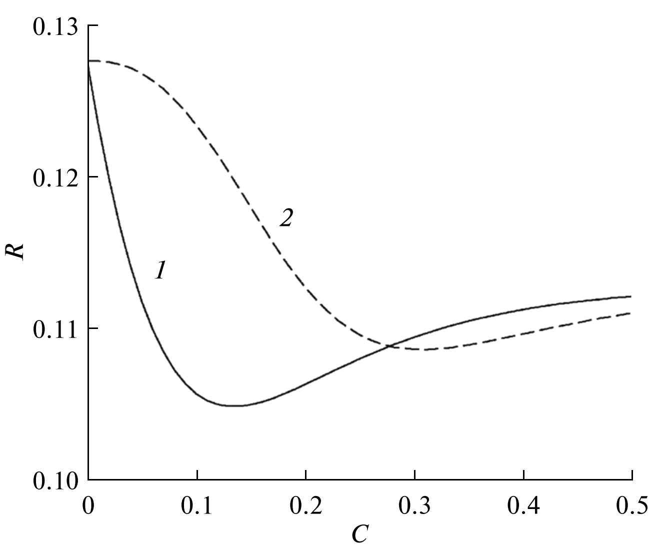

Fig. 5. Dependence of error R on the quantity of threshold level C for the (*1*) soft and (*2*) hard variants of specification of threshold function $p_C(u)$.

Minimum values of error R and optimal threshold level C_{opt} for different values of the noise intensity and two—soft and hard—ways of specification of the threshold function

Noise dispersion	C_{opt} (soft)	R	C_{opt} (hard)	R
0.1	0.134	0.1048	0.303	0.1085
0.2	0.13	0.1864	0.305	0.1884
0.3	0.118	0.2681	0.284	0.2695
0.4	0.1	0.3418	0.258	0.3429
0.5	0.083	0.4023	0.236	0.4031

Thus, the comparative analysis of the results of the wavelet filtering of noisy images makes it possible to conclude that the soft variant of the threshold function specification is advantageous.

CONCLUSIONS

In this work, the problem of improving the quality of digital interference filtering has been studied on the basis of the DVT with the use of various variants of the threshold function specification and wavelet basis choice. Considerable interest to this problem and numerous investigations have made it possible to conclude that the search of optimization ways for suppressing interferences in signals or images continues to be an actual and important problem.

According to the obtained results, the variant of the soft specification of the threshold function has certain advantages, especially, when small values of threshold C are chosen. This threshold provides for the decrease of the filtering error by 15–20% as compared with the hard variant of the threshold function specification. Note that similar conclusions have been made for the analysis of a large number of test images with artificially added interferences. The aim has consisted in obtaining quantitative characteristics indicating the quality of the digital wavelet filtering.

We have also considered the influence of choosing the basis and changing the image dimension on the filtering quality. The image scaling change affects the filtering error value and results in the fact that the error reaches its minimum for different bases. Nevertheless, in many cases, this minimum has been obtained for Daubechies wavelet D^8, which can be considered as the compromise of the wavelet definition range (localization) and regularity of the basis function. The latter allows us to smooth inverse transformation errors that are caused by rejecting small wavelet coefficients. Thus, wavelet D^8 and the soft variant of the threshold function specification should be preferred in practical realization of the digital wavelet filtering methods.

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