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A Method of Evaluating Zero Conditional Lyapunov Exponent from Time Series

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Abstract—A new method of evaluating zero conditional Lyapunov exponent is proposed that is valid for dynamic systems exhibiting periodic dynamics, in the presence of noise, and for coupled chaotic systems. The proposed method is based on analysis of the time series (phase difference versus time) of a system under consideration and allows a zero conditional Lyapunov exponent in a supercritical region of the control parameter to be estimated with sufficiently high accuracy. The main results are illustrated by the example of systems exhibiting periodic dynamics in the presence of noise.

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Studying Lyapunov exponents is a powerful tool for analysis of the behavior of systems with complex dynamics [1–4]. These exponents can be used, in particular, for the identification of a transitions between various regimes (e.g., from periodic or quasi-periodic oscillations to chaotic ones [5] and from chaotic to hyperchaotic ones [6]), establishment of the presence of a hyperbolic attractor [3, 7], and diagnostics of the chaotic synchronization of various types (see, e.g., [8–12]).

As is known, a nonautonomous system exhibiting periodic dynamics is characterized by the presence of a zero (null) conditional Lyapunov exponent. As the coupling parameter (amplitude of external drive action) increases, this Lyapunov exponent passes to a region of negative values, which corresponds to the establishment of synchronization in the system at a critical point ($A = A_c$) [13]. In the case of an additional source of noise acting on this system, the zero Lyapunov exponent becomes negative before the onset of synchronization and its dependence on the parameter of supercriticality (A–A_c) obeys the relation

$$\Lambda_0(\varepsilon) = \begin{cases} \left(-\dfrac{a_1}{|A-A_c|}\right), & A < A_c, \\ f\ln\left|1 - a_2\sqrt{A-A_c}\right|, & A > A_c, \end{cases} \qquad (1)$$

where f is the frequency of the external signal and a_1 and a_2 are constant factors determined by properties of the system [13]. An analogous dependence is characteristic of the zero Lyapunov exponent in a number of systems exhibiting chaotic dynamics (e.g., unidirectionally coupled Rössler oscillators). At the same time, this is not characteristic of some other systems (e.g., the same Rössler oscillators under external harmonic action).

In this Letter, we propose a universal method for evaluating the conditional zero Lyapunov exponent from time series that is valid for systems exhibiting periodic dynamics, in the presence of noise, and for coupled chaotic systems. For nonautonomous periodic oscillators, the results are virtually the same as those provided by the Benettin algorithm [5]. However, in the case of coupled chaotic systems, the proposed method allows the zero Lyapunov exponent in a supercritical region of the control parameter ($A > A_c$) to be determined, whereas the Benettin algorithm gives the value of the highest (i.e., positive) Lyapunov exponent.

Let us illustrate the essence of the proposed method by the example of a quadratic map in the presence of noise:

$$x_{n+1} = f(x_n) + \xi_n = x_n + \Omega x_n^2 - \varepsilon + \xi_n, \qquad (2)$$

where ξ_n is the delta-correlated Gaussian noise [$\langle\xi_n\rangle = 0$, $\langle\xi_n\xi_m\rangle = D\delta(n-m)$], Ω and ε are the control parameters, and D is variance of noise. For the initial system in the absence of noise, a tangent bifurcation takes place at $\varepsilon_c = 0$. As a result, the initially zero Lyapunov exponent passes to the negative region. Let us set the control parameters to be $\Omega = 0.1$ and $\varepsilon = 0.008$ and noise variance $D = 0.0001$ (which corresponds to the supercritical region, $\varepsilon > \varepsilon_c$) and determine the behavior of the Lyapunov exponent in this case.

As was demonstrated in [13], the Lyapunov exponent of this system can be determined using the following formula:

$$\Lambda_0(\varepsilon) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|, \quad (3)$$

where x_n is the time series of system (2) and $f'(x) = 1 + 2\Omega x$ is the analytically calculated derivative of the evolution operator.

Taking into account the ergodicity of the process under consideration and the presence of reinjection, temporal averaging can be replaced by averaging over the ensemble so that

$$\Lambda_0(\varepsilon) = \int_{-\infty}^{\infty}\rho_i(x)\ln|f'(x)|dx, \quad (4)$$

where $\rho_i(x)$ is the distribution density of variable x. As is known, the probability density in the supercritical region of parameters obeys the following relation (for more detail, see [14]):

$$\rho(x) = A\exp\left[-\frac{2}{D}\left(\varepsilon x - \frac{\Omega x^2}{3}\right)\right], \quad (5)$$

where A is the normalization coefficient and D is the variance of noise. Therefore, the values of control parameters Ω and ε can be determined by approximating a numerically calculated distribution by relation (5). Then, the zero Lyapunov exponent can be calculated using the following formula:

$$\Lambda_0(\varepsilon) = \int_{x_1}^{x_2}\rho(x)\ln|1 + 2\Omega x|dx, \quad (6)$$

where the x_1 and x_2 values are empirically determined from the form of $\rho(x)$.

The efficacy of the proposed method is illustrated in Fig. 1, which shows (a) time series x_n of system (2) with the aforementioned control parameters and (b) probability distribution density $\rho(x)$ (histogram) obtained from time series x_n and its approximation (solid curve) by formula (5) with $A = 1.07 \times 10^{-13}$, $\Omega = 0.111124$, $\varepsilon = 0.0087$, and $D = 0.0001$. The values of control parameters were determined using the following procedure. The value of variance D was calculated as the effective phase diffusion coefficient using the method described in [15].

The relation between parameters A, Ω, and ε was determined using the condition of coincidence of the maxima of the numerically calculated probability-distribution density and those of function (5), which led to the following estimations: $\varepsilon = 0.0784\Omega$, $A = 14.2987\exp(-292.693\Omega)$, where the value of Ω was determined by least squares. This procedure leads eventually to the following values of parameters: $\Omega = 0.111124$, $\varepsilon = 0.0087$, and $A = 1.07 \times 10^{-13}$. As can be seen, these estimations are close to the initial values of Ω and ε. Substituting these results into formula (6) and choosing $x_1 = -0.5$ and $x_2 = 0$, we obtain $\Lambda_0 = -0.065$, which agrees well with the zero Lyapunov exponent determined using the Benettin algorithm [5].

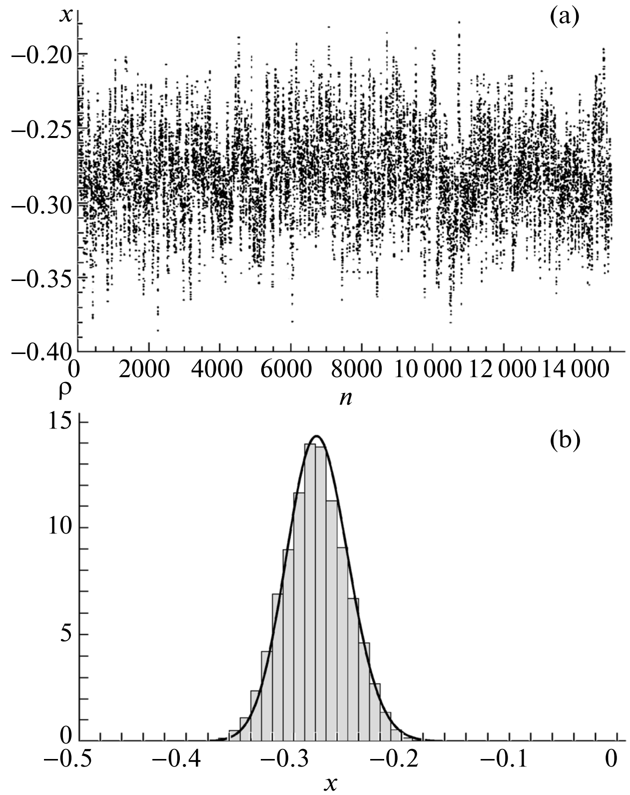

Fig. 1. (a) Time series x_n of quadratic map (2) with $\Omega = 0.1$, $\varepsilon = 0.008$, and $D = 0.0001$ and (b) probability-distribution density $\rho(x)$ (histogram) obtained from time series x_n and its approximation (solid curve) by formula (5) with $A = 1.07 \times 10^{-13}$, $\Omega = 0.111124$, $\varepsilon = 0.0087$, and $D = 0.0001$.

Let us also apply the proposed method to other model systems under the effect of noise. Examples are offered by a system with discrete time, representing the mapping of a circle into itself,

$$x_{n+1} = x_n + 2\Omega(1 - \cos x_n) - \varepsilon + \xi_n, \quad \mathrm{mod}\, 2\pi \quad (7)$$

with $\Omega = 0.1$, and a system with continuous time based on the Adler equation

$$\frac{dx}{dt} = -\Delta + \varepsilon\sin x + \xi \quad (8)$$

with $\Delta = 0.1$ and $D = 1$. Let the noise signal characteristics be the same as those for the quadratic map considered above.

Figure 2 shows the corresponding dependences of the zero Lyapunov exponent on the parameter ε for both model systems. Points show data obtained using the proposed method, while solid curves represent the results of calculations using the Benettin algorithm. As can be seen, the results of the two methods are in good mutual agreement.

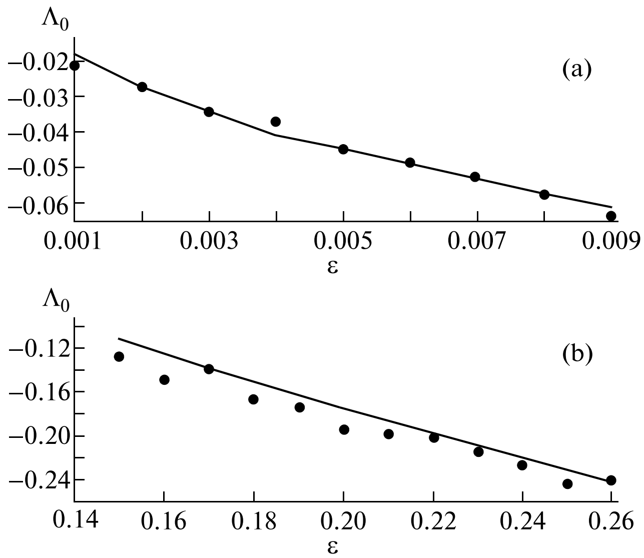

Fig. 2. Dependences of zero Lyapunov exponent Λ^0 on criticality parameter ε for two model systems in the presence of noise: (a) mapping (7) of a circle into itself with $\Omega = 0.1$; (b) Adler equation (8) with $\Delta = 0.1$. Points show data obtained using the proposed method; solid curves represent the results of calculations using the Benettin algorithm.

Thus, we have proposed a new method of evaluating the zero conditional Lyapunov exponent from the time series. The method was verified in for use in modeling one-dimensional systems with continuous and discrete time (mapping of a circle into itself and Adler equation) in the presence of noise. It was shown that the proposed method is effective in the supercritical region of control parameters. It can be expected that this method is universal and will also provide correct results in application to calculations of the zero conditional Lyapunov exponent for nonautonomous and coupled systems exhibiting chaotic and periodic dynamics (in the presence of noise). The proposed method can also be used for determining the degree of synchronism between time series generated by systems of different natures.

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