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Using the Spectrum of Lyapunov Exponents to Analyze the Dynamics of Beam–Plasma Systems Simulated by the Large Particle Method

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Abstract—A method is proposed for calculating the spectrum of Lyapunov exponents for spatially distributed beam–plasma systems simulated by the large particle method. The effectiveness of the proposed approach is demonstrated by the example of an electron reference system consisting of a Pierce diode and a model of a low-voltage vircator as a source of broad-band microwave radiation.

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INTRODUCTION

Lyapunov exponents are a powerful mathematical tool for analyzing the complex nonlinear dynamics of systems with different natures [1]. However, effective use of this tool is restricted mainly to systems with few degrees of freedom [2]. At the same time, a wide class of real systems and objects can be described using spatially distributed dynamic variables. Spatially distributed models are widely used in particular for analyzing processes that occur in the components and devices of vacuum, plasma, and semiconductor electronics in the microwave and THz ranges [3–5]. The problem with applying Lyapunov exponents is in this case due to the existing calculation algorithms used for systems with few degrees of freedom being inapplicable for analyzing systems with infinite dimensional phase space. The algorithm used in [6] to calculate Lyapunov exponents for spatially distributed beam–plasma systems is restricted only to models of the continuous medium type (e.g., to hydrodynamic descriptions of the electron flux), preventing us from analyzing the spectra of Lyapunov exponents for a wide class of systems described by particle methods. When simulating such systems in nonlinear modes of operation, the overtaking and reflection of particles are typically observed, rendering hydrodynamic descriptions ineffective. At the same time, the above are an important class of electron and plasma systems, i.e., electromagnetic radiation amplifiers and generators that exhibit a wide range of nonlinear processes, making it of interest to study the quantitative characteristics of their nonlinear dynamics [7–10].

In this work, a method for calculating the spectrum of Lyapunov exponents for spatially distributed beam–plasma systems simulated using the large particle method is proposed for the first time. The effectiveness of the proposed approach is demonstrated by the example of an electron reference system (a Pierce diode) and a model of a promising source of broadband microwave radiation (a low-voltage vircator) [3, 12].

INVESTIGATED SYSTEM AND NUMERICAL MODEL

The Pierce diode chosen as our main object of study is one of the simplest electron–plasma models. Despite this, the system is capable of producing different nonlinear phenomena, including random oscillations and the formation of structures [7, 11, 13]. The Pierce diode is an interaction space formed by two ground grid electrodes; a monoenergetic beam of electrons is injected into the space at the input. The space between the grids is filled with a neutralizing ion gas whose density is equal to the nondisturbed density of the electron flux charge.

In this work, the dynamics of an electron beam in the Pierce diode in the mode of forming a virtual cathode (VC) is considered. It is impossible to describe the electron flux in the context of the hydrodynamic model, due to the reflection of particles inside the flux. When studying the complex nonlinear dynamics of oscillations of a nonstationary virtual cathode, numerical simulations using the particle-in-cell method [14] are therefore used. In dimensionless variables, the mathematical model based on the particle method has the form [15–17]

$$\frac{\partial^2 x_i}{\partial t^2} = -E(x_i), \quad (1)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \alpha^2(\rho(x_i) - \rho_{\text{ion}}), \quad (2)$$

where E is the field strength, x_i is the coordinate of the ith charged sheet, φ is the potential of the spatial charge field, $\rho(x_j)$ is the density of the spatial charge in a node with coordinate x_j, ρ_{ion} is the density of the neutralizing ion gas, and $\alpha = \omega_p L/\upsilon_0$ is the Pierce parameter (where L is the length of the intergrid space and υ_0 is the unperturbed electron velocity). Equation (2) is complemented by boundary conditions for the potential at grids of the Pierce diode $\varphi(x = 0) = \varphi(x = 1) = 0$.

In a classical Pierce diode, $\rho_{\text{ion}} = 1$, and this is the one that we consider in this work. In the context of the above model, it is also possible to describe the dynamics of a low-voltage vircator—a vacuum device of microwave electronics in which the VC is formed due both to the action of spatial charge forces and to additional deceleration of the beam in the drift space [3, 16, 18, 19]. The simplest model of a low-voltage vircator is also described by Eqs. (1)–(3), but there is no ion background ($\rho_{\text{ion}} = 0$) and the boundary conditions for the potential are modified in the form $\varphi(x = 0) = \varphi(x = 1) = -\Delta\varphi$, where $\Delta\varphi$ is the difference between decelerating potentials. In this work, we consider the dynamics of the spectrum of Lyapunov exponents in the low-voltage vircator model as our second example.

METHOD FOR CALCULATING LYAPUNOV EXPONENTS

It is well known that when calculating the spectrum of Lyapunov exponents, it is necessary to consider the set of small perturbations of the ground state of the system in addition to the dynamics of this state. Applying the approaches used earlier in analyzing the hydrodynamic model of beam–plasma systems [6, 20] is in this case difficult, since the state of system is characterized both by parameters continuously distributed in its space and by a set of discrete particles. The main difficulties are associated with specifying a perturbation and obtaining linearized equations that describe the dynamics of a discrete set of particles. At the same time, we can calculate linear perturbations from their variations using equations that describe the beam in the context of the continuous medium model [21]. Let us consider this approach in more detail.

As in [6], we choose a set of spatial–temporal distributions $U(x,t) = (\rho(x,t), \upsilon(x,t))^T$ for the state of the system. To calculate the initial N exponents, we consider a set of perturbations in density and velocities $V_i(x,t) = (\xi_i^{\upsilon}(x,t), \xi_i^{\rho}(x,t))^T$, $i = 1, \ldots, N$, satisfying the conditions of normalization and orthogonality. Such a set of perturbations can be obtained by means of standard Gram–Schmidt orthogonalization [1]. Note that the distribution of the potential and its perturbations are excluded from the states of the system, since these parameters can be uniquely expressed via the charge density by using the Poisson equation [6].

As was mentioned above, one feature of the method is using a system of continuum equations (hydrodynamic equations) linearized in the neighborhood of the state $U(x, t)$ to describe the dynamics in time and perturbation space. The joint solution to the initial equations of model (1)–(3) for finding the reference state and linear hydrodynamic equations for perturbations allows us to estimate the spectrum of Lyapunov exponents. To accomplish this, the set of perturbations is subjected to Gram–Schmidt orthogonalization after some period T chosen empirically. This process is repeated M times, and sums $S_i = \sum_{j=1}^{M} \ln\|\tilde{V}_i(x, jT)\|$, where $\tilde{V}_i(x, jT)$ is the distribution of the ith perturbation before renormalization but after orthogonalization. The values of the Lyapunov exponents are determined as [6]

$$\Lambda_i = \frac{S_i}{MT}. \quad (3)$$

Applying the above approach to analyzing the electron flux dynamics in our systems demonstrated the high efficiency of this method in obtaining a quantitative estimate of the random behavior of spatially distributed beam–plasma systems.

RESULTS FROM CALCULATING THE SPECTRUM OF LYAPUNOV EXPONENTS

Let us consider our results from calculating the spectrum of Lyapunov exponents for the investigated model systems. We begin with the Pierce diode, for which it is well known that a VC is formed in the system in the range of the controlling Pierce parameter $\alpha = [\pi, 2\pi]$ and the beam dynamic is characterized by complex, irregular oscillations over the range of controlling parameter variation. The Lyapunov exponent spectrum constructed for this case (Fig. 1) contains positive exponents, corresponding to random dynamics. It is interesting that the hyperchaos mode is observed in the system at $\alpha < 4.7$, as is indicated by the two positive Lyapunov exponents. This is in good agreement with the established results: the complexity of oscillations in the Pierce diode declines with an increase in the Pierce parameter.

The results from calculating the spectrum of Lyapunov exponents for a low-voltage vircator with varying decelerating potential and fixed Pierce param-

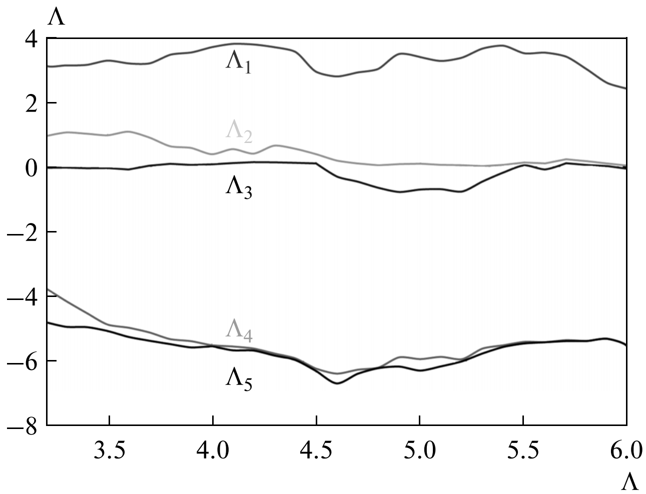

Fig. 1. Five senior Lyapunov exponents as functions of controlling parameter α for the Pierce diode.

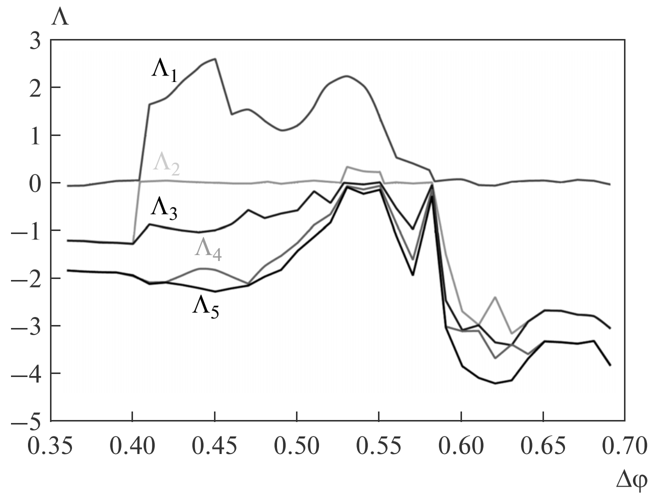

Fig. 2. Five senior Lyapunov exponents as functions of controlling parameter Δφ for the low-voltage vircator model, constructed at $\alpha = 0.9$.

eter $\alpha < 0.9$ are presented in Fig. 2. The obtained spectrum of Lyapunov exponents reflects the behavior of the system with variation in the decelerating difference of potentials. It can be seen that the electron beam exhibits both periodic ($0.36 < \Delta\varphi < 0.41$ and $0.58 < \Delta\varphi < 0.69$) and random ($0.4 < \Delta\varphi < 0.58$) dynamics with variations in the decelerating field in the interaction space. In the region of $0.53 < \Delta\varphi < 0.55$, the system transitions to the hyperchaos mode, as is indicated by the two positive exponents in the spectrum. Our results from calculating the spectrum of Lyapunov exponents agree well with the other methods for theoretical and experimental analysis of vircator dynamics presented in [3, 16, 18, 19, 22]. The hyperchaos mode corresponds in particular to the most complicated oscillation mode in the system; periodical modes were also observed at low and high decelerating potentials.

CONCLUSIONS

The question of calculating the spectrum of Lyapunov exponents for spatially distributed beam–plasma systems simulated by the large particle (particle-in-cell) method was examined in this work. An electron reference system consisting of a Pierce diode in the mode of VC formation and a low-voltage vircator were chosen for our objects of study. Even though the behavior of such a system can be described by the large particle method, the dynamics of small perturbations of the ground state can be simulated using hydrodynamic equations linearized in a neighborhood of the reference path. In simulating the dynamics of the system jointly with small perturbations and applying Gram–Schmidt orthogonalization, the spectra of Lyapunov exponents were thus calculated for model systems. It was shown for the first time that, depending on the magnitude of the decelerating potential, the considered system can exhibit both periodic dynamics and complex random modes that include hyperchaos.

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