ISSN 1062-8738, Bulletin of the Russian Academy of Sciences. Physics, 2014, Vol. 78, No. 12, pp. 1324–1327. © Allerton Press, Inc., 2014.
Original Russian Text © A.A. Kharchenko, V.V. Makarov, A.E. Khramov, 2014, published in Izvestiya Rossiiskoi Akademii Nauk. Seriya Fizicheskaya, 2014, Vol. 78, No. 12, pp. 1616–1620.

Studying Synchronization in a Network of Nonlinear Oscillators with a Complex Topology of Relations, According to Integral Registered Characteristics

A. A. Kharchenko[a], V. V. Makarov[a, b], and A. E. Khramov[a, b]
[a] *Faculty of Nonlinear Processes, Saratov State University, 410012 Russia*
[b] *Saratov State Technical University, 410054 Russia*
e-mail: ainadil@mail.ru; hramovae@gmail.com

Abstract—A numerical study of a complex network of coupled (Van der Pol) oscillators is performed. The problem of analyzing phase synchronization in a network by studying the index of phase synchronization, calculated both along all elements of the network and according to the integral characteristics at different frequencies of generator mismatch and different topologies of the bonds between network elements.

DOI: 10.3103/S1062873814120120

INTRODUCTION

Investigating dynamic processes in networks of nonlinear elements with complex topologies of relations is today one of the most important fields of research in radiophysics and nonlinear dynamics [1, 2]. Such studies are of special importance to neurodynamics and neurophysiology, where the investigated objects are neural ensembles of the central and peripheral nervous systems, which are complex networks of elements, i.e., neurons, with their own complex dynamics [3]. Recording electroencephalograms (EEGs), the averaged sums of electrical fields generated by synaptic currents in major groups of neurons in the vicinity of recording microelectrodes, is a conventional and quite effective method for studying the electrical activity of the brain. It is assumed that each neuron generates pulses with the same shape and amplitude, the pulse shape being unique to a particular neuron and the amplitude depending on the distance from a cell to the microelectrode [4]. To record the EEGs of a human brain, electrodes are typically placed on the scalp, while implanted electrodes, which enable us to obtain more detailed electrical activity data on relatively small populations of neurons in the cerebral cortex and subcortical structures, are conventionally used for animals. We may thus conclude that EEG signals are averaged (integral) characteristics that describe the dynamics of complex neural networks. Rising EEG amplitude indicates an increase in the coherence of oscillations in a neural ensemble in the vicinity of the recording electrode. There have recently been growing efforts to use EEG signals for diagnostics of various types of synchronization in neural ensembles, posing the important question of the need to estimate the efficiency of integral characteristics, specifically ones averaged by a neural ensemble.

The aim of this work was to study a network of coupled oscillators and compare the quality of diagnostics of its synchronization in terms of the signals of individual oscillators and the integral characteristics of the ensemble.

MODEL AND METHOD FOR ANALYZING SYNCHRONIZATION IN A NETWORK

Our object of study was a network of oscillators; specifically, dissipatively coupled Van der Pol oscillators:

$$\ddot{x}_i - \mu(1 - x_i^2)\dot{x}_i + \delta_i^2 x_i = \varepsilon \sum_{j=1}^{N} c_{ij}(x_i - x_j), \qquad (1)$$

Here, i and j indicate the number of the element in the network; N is the number of elements in the network; μ is a coefficient that characterizes the nonlinearity and damping rate of oscillations; ε is a parameter that indicates coupling between oscillators; and δ_i is the angular frequency of an individual oscillator.

Coefficients c_{ij} of the coupling matrix determine the network topology. If the oscillator with number i affects the oscillator with number j, then $c_{ij} = 1$. If oscillators do not interact (i.e., there is no coupling), then $c_{ij} = 0$. We shall consider a symmetric coupling matrix characterized by expression $c_{ij} = c_{ji}$. (From a physical perspective, this implies that only bidirectional couplings take place in the system.) In addition, couplings are defined as dissipative, imposing the following constraint on the elements of the coupling matrix lying on the main diagonal:

$$c_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij}. \tag{2}$$

It is worth noting that the values of coefficients c_{ij} indicate merely the presence of coupling, while the intensity of interaction is determined by parameter ε, referred to as the coupling coefficient or coupling intensity [5].

This system is of great interest, since the Van der Pol oscillator is a fundamental model of nonlinear theory and is often used as a basic model for investigating the phenomenon of phase synchronization. The concept of phase synchronization relies on the concept of the instantaneous phase $\varphi(t)$ of a signal. The attractor of the system must be such that the phase trajectory projection onto a certain plane of states $(x, \dot{x})$ rotates continuously around the origin of coordinates without either intersecting or enveloping it (a phase-coherent attractor) [6]. Instantaneous phase $\varphi(t)$ of the signal can then be introduced as an angle in polar coordinates in the plane $(x, \dot{x})$,

$$\varphi(t) = \arctan\left(\frac{\dot{x}}{x}\right). \tag{3}$$

Phase synchronization behavior implies that phases of the signals of interacting oscillators are captured; i.e., their difference does not rise in absolute value over time. In other words, it does not exceed a prescribed constant; 2π is usually considered [6]:

$$|\Delta\varphi_{ij}(t)| = |\varphi_i(t) - \varphi_j(t)| < 2\pi. \tag{4}$$

To quantitatively characterize the degree of synchronization in the network, we consider its synchronization index [5, 7], which is a value proportional to the number of synchronized oscillators:

$$\sigma = \frac{2}{TN^2}\left|\sum_{i=1}^{N}\sum_{j=i+1}^{N} e^{\sqrt{-1}\Delta\varphi_{ij}(t)}\right|, \tag{5}$$

Here, $\Delta\varphi_{ij}$ is the phase difference between the ith and jth oscillators in the network; T is the duration of temporal implementation in which the processes in the network were simulated. Synchronization index values close to zero indicate that very few oscillators are synchronized, while an increase in σ shows that growing numbers of oscillators reach synchronization in the network (cluster synchronization). Complete phase synchronization is reached in the domain under study at $\sigma = 1$.

To analyze synchronization according to integral characteristics, we introduce the following functions (essentially oscillation characteristics averaged over a certain subset M of all elements of the network):

$$X_k(t) = \frac{1}{M}\sum_{n=k}^{k+M} x_n(t), \tag{6}$$

Here, M is the number of elements according which the averaging is performed, and k is the number of the first element of the subset.

In the first approximation, (6) may be considered an analog of the integral EEG signal that determines the contribution from a certain group of oscillators (neurons from the local domain of the neural network that are close to the recording electrode) to the experimentally recorded signal of the oscillator network (1).

To analyze synchronization according to integral characteristics (6), we also use an approach based on calculating synchronization index (5), but integral averaged characteristics $X_k(t)$ are used as signals for phase recovery.

RESULTS FROM ANALYZING PHASE SYNCHRONIZATION ON THE BASIS OF INTEGRAL CHARACTERISTICS

Let us consider the results from numerical simulations of a network of coupled Van der Pol oscillators (1). A network of 100 oscillators is considered, and μ_i values that characterize nonlinearity we take to be constant; specifically, $\mu = 1.0$ for all oscillators. The coupling parameter of the oscillators is varied within the interval $0 \le \varepsilon \le 1$ with constant step $\Delta = 0.01$. The frequency detuning of oscillators in the network is set as a uniform distribution within the frequency range $\delta_i \in [0.2, 0.4]$.

Two different approaches to constructing the oscillator network topology (i.e., two different topologies of relations) were considered in our investigation of the system. The first type of network was random with arbitrary relations between the elements of the network and a variable average number of relations per oscillator (coupling concentration $\beta = \bar{n}/N$ where $\bar{n}$ is the average number of relations per one element of the network, and N is the total number of elements in the network). The second type of network was a small-world model [1, 2], constructed such that each network node was related to five nearby oscillators, and one of the short link connections was randomly replaced by a long one. In the latter case, the so-called small worlds phenomenon was observed. This network design has a high degree of clusterization [1].

Let us first consider the network with random topology of relations. The dependence of synchronization index on coupling intensity ε at different network parameters β is presented in Fig. 1 (curves *1* and *2*). The synchronization index was based on all elements of the network. Several characteristic regions can be distin-

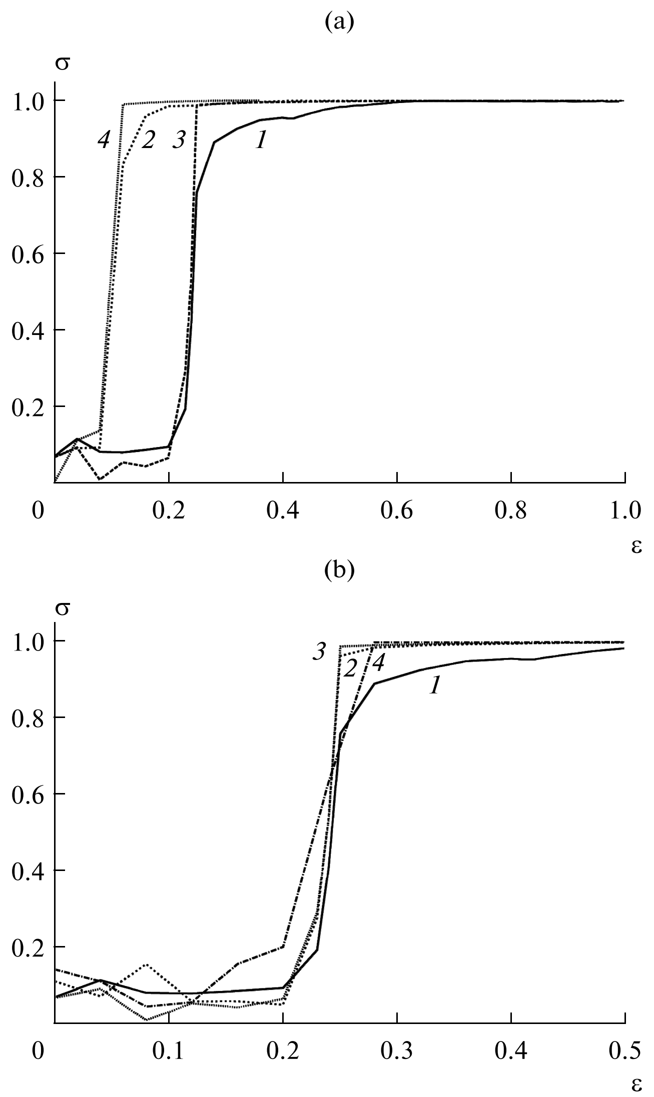

Fig. 1. (a) Dependence of the synchronization index on the coupling coefficient for oscillators. Curves *1* and *2* correspond to $\beta = 0.05$; curves *3* and *4* correspond to $\beta = 0.07$. Dependences *3* and *4* were constructed using the integral characteristics for $M = 10$; (b) Dependence of the synchronization index on the coupling coefficient for oscillators using the integral characteristics for different M: (*1*) $M = 1$, (*2*) $M = 4$, (*3*) $M = 10$, (*4*) $M = 20$; $\beta = 0.05$.

guished in terms of the behavior of the system. A region with the low synchronization index corresponding to $\sigma \sim 0.1$ is observed for low coupling coefficients. This region corresponds to the nonsynchronous behavior of the system, but its size varies considerably, depending on the average number of relations. The first region corresponds to the coupling parameter $\varepsilon \in [0: 0.2)$ for $\beta = 0.05$, and its shrinks greatly ($\varepsilon \in [0: 0.08)$) for $\beta = 0.07$.

In addition, a considerable increase in σ is clearly seen in the synchronization index dependence, which grows to 1. Synchronous behavior is established in this region through cluster synchronization, when several synchronous clusters that include certain subsets of elements of the network are formed in it. Finally, the third region, in which the synchronization index equals 1, corresponds to complete phase synchronization of the network, when all the elements of the network are synchronized with one another. It is clearly seen that complete phase synchronization for $\beta = 0.05$ is reached at $\sigma \sim 0.6$; for $\beta = 0.07$, it is reached at $\sigma \sim 0.35$.

Let us consider the diagnostics of transitions to phase synchronization using integral characteristics (6). The corresponding curves for $\beta = 0.05$ and $\beta = 0.07$ with $M = 10$ are presented in Fig. 1 (curves *3* and *4* respectively). It is clearly seen that the use of integral characteristics leads to significantly earlier diagnostics of the boundary of transition to complete phase synchronization when we use the coupling parameter than when analyzing the dynamics of each individual element of the network. It is clearly seen that complete phase synchronization is identified at $\varepsilon \sim 0.22$ for $\beta = 0.05$ and at $\varepsilon \sim 0.17$ for $\beta = 0.07$ when this type of diagnostics is used. This is because when a large number of elements reach synchronization, the use of averaged characteristics (6) makes it impossible to specify nonsynchronous elements; they become indistinguishable as a result of averaging, thereby allowing us to lower synchronization index (5). The use of integral characteristics in fact does not allow us to analyze the transition to complete phase synchronization based on the cluster synchronization behavior of the network. We therefore conclude that when the diagnostics of synchronization is performed in neural ensembles by recording the observed EEG or magnetoencephalogram signals [4, 8], the boundary of establishing phase synchronization in a neural ensemble will always be underestimated.

The dependence of synchronization indices for different M values of integral characteristic (6) is presented in Fig. 1b. It is clearly seen that raising M (i.e., increasing the number of elements with which averaging is performed) steepens the synchronization index function and thus results in poorer accuracy when determining the phase synchronization boundary in a network.

Let us consider a more complex network with a small-world topology of relations. It is known that the set of neural subnetworks in the brain are organized into structures close to the small-world structure in [9]. Synchronization indices constructed on the basis of all elements in a network and on the integral characteristics of the coupling parameter for a small-world network are shown in Fig. 2. It is clearly seen that the difference between the boundaries of establishing complete phase synchronization in network (1), analyzed using integral characteristics (6) and based on individual elements of the network, is not as great.

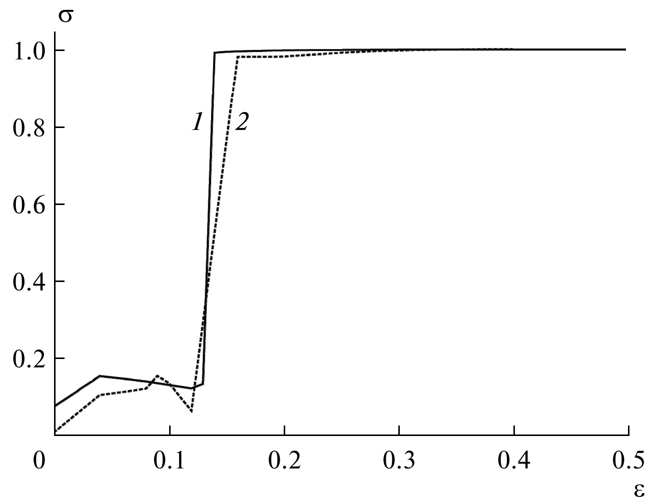

Fig. 2. Dependence of the synchronization index on the coupling coefficient for oscillators in the network with the small-world topology of relations. Curve *1* corresponds to a synchronization index based on individual elements of the network; curve *2* was constructed using the integral characteristics for $M = 10$.

Using the method based on observing the integral characteristics thus yields better results for a more structured network, since developed cluster synchronization is not observed in this type of network.

CONCLUSIONS

Networks of coupled oscillators with different topologies of the relations between elements (small-world network and random network) were investigated. Quantitative analysis of phase synchronization was performed by calculating the phase synchronization indices for all elements of network (1) and based on integral characteristics (6) at different frequency detuning values of the oscillators, and for different topologies of the relations between elements of the network. It was shown that diagnostics of the synchronization based on integral characteristics provides a correct description of the processes at high coupling intensities, when complete synchronization is achieved throughout a network. On the other hand, the use of integral characteristics at low coupling coefficients can lead to errors in identifying the boundaries of the establishing of synchronous dynamics, particularly the formation of synchronous clusters.

ACKNOWLEDGMENTS

We thank A.A. Koronovskii and V.V. Grubov for their helpful comments on this work.

This work was supported by the RF Ministry of Education and Science, State Program for Research at Institutions of Higher Education, 2014–2016; by the Russian Foundation for Basic Research, project nos. 12-02-00221 and 14-02-31235; and by the RF Presidential Program for the Support of Young Doctors of Science, MD-345.2013.2.

REFERENCES

1. Strogatz, S.H., *Nature*, 2001, vol. 410, pp. 268–276.
2. Boccaletti, S., et al., *Phys. Rep.*, 2006, vol. 424, pp. 175–308.
3. Buzsaki, G. and Draguhn, A., *Science*, 2004, vol. 304, 1
pp. 1926–1929.
4. Koronovskii, A.A., Makarov, V.A., Pavlov, A.N., Sitnikova, E.Yu., and Khramov, A.E., *Veivlety v neirodinamike i neirofiziologii* (Wavelets in Neurodynamics and Neurophysiology), Moscow: Fizmatlit, 2013.
5. Moskalenko, O.I., Phrolov, N.S., Koronovskii, A.A., et al., *Eur. Phys. J. Spec. Top.,* 2013, vol. 222, pp. 2571–2582.
6. Koronovskii, A.A., Kurovskaya, M.K., Moskalenko, O.I., et al., *Zh. Tekh. Fiz.*, 2007, vol. 77, pp. 21–29.
7. Tass, P., Rosenblum, M.G., Weule, J., et al., *Phys. Rev. Lett.*, 1998, vol. 81, pp. 3291–3294.
8. Pavlov, A.N., Khramov, A.E., Koronovskii, A.A., Sitnikova, E.Yu., *Usp. Fiz. Nauk*, 2012, vol. 182, no. 9, pp. 905–930.
9. Li, C., Sun, W., and Kurths, J., *Phys. Rev. E*, 2007, vol. 76, p. 046204.

Translated by A. Amitin

SPELL: 1. ok