Inapplicability of an auxiliary-system approach to chaotic oscillators with mutual-type coupling and complex networks

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The auxiliary system approach being *de facto* the standard for the study of generalized synchronization in the unidirectionally coupled chaotic oscillators is also widely used to examine the mutually coupled systems and networks of nonlinear elements with the complex topology of links between nodes. In this Brief Report we illustrate by two simple counterexamples that the auxiliary-system approach gives incorrect results for the mutually coupled oscillators and therefore to study the generalized synchronization this approach may be used only for the drive-response configuration of nonlinear oscillators and networks.

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Generalized synchronization is an intricate fundamental phenomenon of nonlinear sciences. The notion of generalized synchronization was introduced initially for two unidirectionally coupled dynamical systems demonstrating chaotic behavior [1], with numerous numerical and experimental examples demonstrating such phenomena being found [2–7].

To detect the generalized synchronization regime in unidirectionally coupled systems, different techniques have been proposed, e.g., the nearest-neighbor method [1,8] or the conditional Lyapunov exponent calculation [9]. Among these techniques the auxiliary system approach proposed initially for unidirectionally coupled chaotic oscillators may be generally considered as the most easy, clear, and powerful tool to study the generalized synchronization regime in chaotic systems. Starting from the seminal paper of Abarbanel *et al.* [10], the auxiliary system approach has become *de facto* the standard of generalized synchronization studies being used in many of the theoretical and experimental works (see, e.g., [3,11–14]).

In parallel with the unidirectionally coupled systems, the auxiliary-system approach (usually without the proper theoretical justification) was applied to the mutually coupled oscillators [15] and the networks with a complex topology of links between nodes [16,17]. Now this approach is used widely as the standard tool to detect generalized synchronization for the oscillators and networks with a mutual type of coupling [18–22].

In this Brief Report we prove that the application of the auxiliary-system approach to systems with a mutual type of coupling for the generalized synchronization threshold detection is misleading and leads to incorrect results even for two mutually coupled oscillators. This aspect is critically important since both generalized synchronization and complex networks are subjects of great interest of scientists and the use of incorrect tools may leads to misleading results in the topical branches of knowledge.

The modification of the auxiliary-system approach proposed in [15] is developed for two mutually coupled chaotic oscillators 1 and 2. Along with the original systems, two additional auxiliary units 1′ and 2′ coupled unidirectionally with 2 and 1, respectively, are treated (see Fig. 1 for details). The systems 1 and 1′ (as well as 2 and 2′) are characterized by the same control parameter values, but evolve with different initial conditions lying in the same basin of attraction. When only one couple of systems (say, 2 and 2′) starts demonstrating identical behavior, the presence of the partial generalized synchronization (PGS) regime is assumed to take place. As soon as all sets of two oscillators show identical behavior in pairs it is usual to detect the global (or complete) generalized synchronization (CGS) regime.

At first sight, this extension of the auxiliary-system approach to the mutually coupled systems seems to be reasonable and true. Nevertheless, after further examination we arrived at the conclusion that the results obtained by means of this method are incorrect.

To show the failure of both the proposed concept of the partial and global generalized synchronization and the auxiliary-system method extension to the systems with bidirectional coupling we consider first two mutually coupled Rössler oscillators, which are excellent model systems to be used as the counterexample. Particularly two coupled Rössler systems are well known to show the transition from the asynchronous dynamics through the phase synchronization regime to lag synchronization when the coupling strength between them grows [23]. At the same time, lag synchronization is known to be a special case (moreover, the strong form) of the generalized synchronization [9] since in the case of the lag synchronization regime the functional relation $\mathbf{x}_2(t) = \mathbf{x}_1(t-\tau)$ is certain to exist, which may be verified, e.g., by the phase tube approach [24]. In other words, the presence of the lag synchronization regime in the interacting systems is irrefutable evidence of the existence of generalized synchronization and we use it in our counterexample.

The equations describing two mutually coupled Rössler systems are

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),\\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2},\\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c),\end{aligned} \qquad (1)$$

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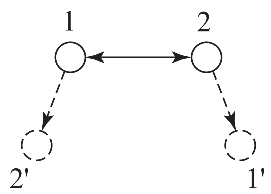

FIG. 1. Schematic representation of the extension of the auxiliary-system method to the mutually coupled oscillators proposed in [15].

where $\mathbf{x}_{1,2}(t) = (x_{1,2}, y_{1,2}, z_{1,2})^T$ are the vector states of the interacting systems, $a = 0.15$, $p = 0.2$, and $c = 10$ are the control parameters, and ε is a coupling parameter. The parameter $\omega_{1,2}$ defines the natural frequency of oscillations. In our studies we have varied the ω_1 parameter providing the frequency mismatch of the oscillators, with $\omega_2 = 0.95$ being fixed.

In Fig. 2 the onset of the lag synchronization regime and PGS and CGS boundaries detected with the help of the extended auxiliary-system method are shown. To find the PGS and CGS boundaries (as well as the generalized synchronization onset in the case of the unidirectionally coupled systems described below) we have computed the average distances

$$D_i = \lim_{T\to\infty} \frac{1}{T - T_0} \int_{T_0}^{T} \|\mathbf{x}_i(t) - \mathbf{x}'_i(t)\| dt \qquad (2)$$

between the original $\mathbf{x}_i(t)$ and auxiliary $\mathbf{x}'_i(t)$ systems for different values of the coupling strength ε. In (2) $\|\mathbf{x}\| = \sqrt{x^2 + y^2 + z^2}$, i is the number of considered oscillators ($i = 1,2$ for the case of mutual coupling and $i = 2$ for the unidirectionally coupled systems), $T = 2 \times 10^4$ is the time of the calculation, and $T_0 = 10^5$ is the transient. When D_i approaches zero, the state vectors of the original (response for the unidirectionally coupled oscillators) and auxiliary systems begin to coincide with each other, which means the presence of partial ($D_i = 0$ for $i = 1$ or 2) or complete ($D_i = 0$ for both $i = 1$ and 2) generalized synchronization (generalized synchronization for the case of the unidirectional coupling). The obtained boundaries have also been verified by the conditional Lyapunov exponent calculation as well as with the help of the nearest-neighbor method. We have also verified that the increase of parameters T_0 and T does not change the boundary points.

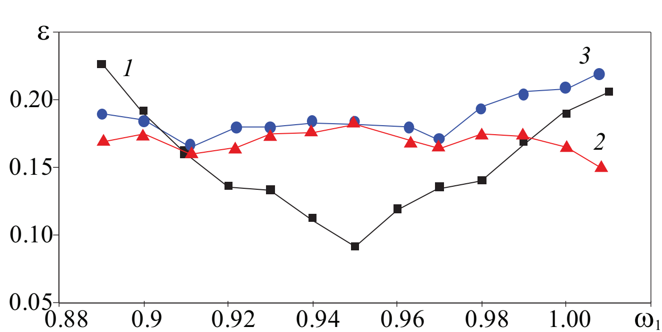

FIG. 2. (Color online) Parameter plane of two mutually coupled Rössler oscillators (1): the onset of the lag synchronization regime (curve 1) and the boundaries of the PGS (curve 2) and CGS (curve 3) regimes found with the help of the modified auxiliary-system approach proposed in [15].

To detect the lag synchronization regime and find its boundaries we have computed (in the case of both the unidirectional and mutual coupling) the dependences of the minimum of the similarity function [23] $\sigma = \min_\tau S(\tau)$ on the coupling parameter value, where

$$S^2(\tau) = \frac{\langle [x_2(t+\tau) - x_1(t)]^2 \rangle}{\sqrt{\langle x_1^2(t)\rangle \langle x_2^2(t)\rangle}}. \qquad (3)$$

Additionally, based on the fact that the lag synchronization regime in flow systems corresponds to the complete synchronization in discrete maps obtained from the initial flow oscillators with the help of the Poincaré secant [25], we have analyzed the complete synchronization regime onset in maps obtained in such a way to verify the found boundaries.

One can see that almost in the whole considered range of the ω_1-parameter values the lag synchronization regime is observed sufficiently below both PGS and CGS boundaries. In other words, according to the concept proposed in [15] (and used later in [16,17]) neither PGS nor CGS is observed when in fact lag synchronization already exists. There is no doubt that the lag synchronization detection without generalized synchronization is misleading. It contradicts the definition of the generalized synchronization regime itself since the complete and lag synchronization regimes are partial cases of generalized synchronization and correspond to its strong form [9]. Obviously, such results must be rejected as erroneous.

As far as two unidirectionally coupled Rössler oscillators

$$\begin{aligned} &\dot{x}_1 = -\omega_1 y_1 - z_1, \quad \dot{y}_1 = \omega_1 x_1 + a y_1, \\ &\dot{z}_1 = p + z_1(x_1 - c), \quad \dot{x}_2 = -\omega_2 y_2 - z_2 + \varepsilon(x_1 - x_2), \\ &\dot{y}_2 = \omega_2 x_2 + a y_2, \quad \dot{z}_2 = p + z_2(x_2 - c) \end{aligned} \qquad (4)$$

are concerned, the boundary of the lag synchronization regime is observed above the generalized synchronization onset (see Fig. 3) in the whole plane of the control parameters and therefore there is no contradiction between the definition of generalized synchronization and the location of the synchronous regimes on the parameter plane.[1]

The cause of the failure of the auxiliary-system method in the case of the mutual type of coupling is the hidden nonequivalence of the original oscillators 1 and 2 and their auxiliary replicas 1′ and 2′ determined by the topology of coupling links between systems under study. Although both systems 2 and 2′ are driven by the same signal 1, the original oscillator 2 also acts on the second oscillator 1, whereas the auxiliary

[1]The counterintuitive behavior of the generalized synchronization onset when the critical coupling strength becomes maximum at $\omega_1 = \omega_2$ (the same effect can be seen for PGS and CGS in Fig. 2) was originally reported in the work of Zheng and Hu [4] and was explained later with the help of the modified system approach in our work [26].

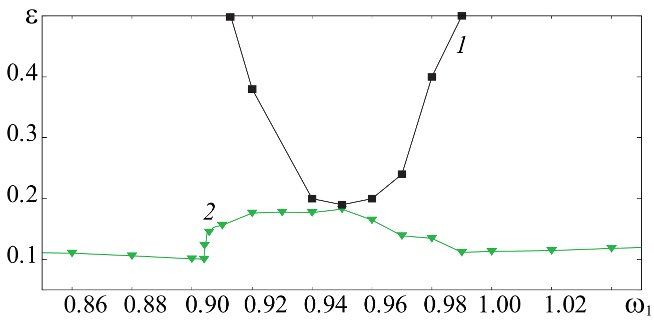

FIG. 3. (Color online) Parameter plane of two unidirectionally coupled Rössler oscillators (4): the onset of the lag synchronizarion regime (curve 1) and the boundary of the generalized synchronization regime (curve 2) found with the help of the auxiliary-system approach [10].

replica $2'$ does not. In this case there is some kind of feedback between the original oscillators 2 and 1, which is absent for $2'$ (and $1'$). Driving oscillator 2, the second original system 1 in turn adjusts to its dynamics (contrary to the dynamics of the auxiliary system $2'$). Obviously, two mutually coupled original systems 1 and 2 become synchronized sufficiently early in comparison with the unidirectionally coupled oscillators 1 and $2'$ (as well as 2 and $1'$). Therefore, the auxiliary-system method applied to the mutually coupled oscillators detects supposedly only the generalized synchronization onset when in fact the oscillators under study are already synchronized greatly as may be seen easily in Fig. 2.

Thus having considered two bidirectionally coupled Rössler oscillators we arrive at the conclusion that the auxiliary-system approach cannot be used correctly to detect the generalized synchronization regime in both the oscillators and networks of nonlinear elements with the mutual type of coupling. The very same conclusion can be made for two bidirectionally coupled Lorenz systems

$$\begin{aligned}\dot{x}_{1,2} &= \sigma(y_{1,2} - x_{1,2}) + \varepsilon(x_{2,1} - x_{1,2}),\\ \dot{y}_{1,2} &= r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2},\\ \dot{z}_{1,2} &= -bz_{1,2} + x_{1,2}y_{1,2}\end{aligned} \qquad (5)$$

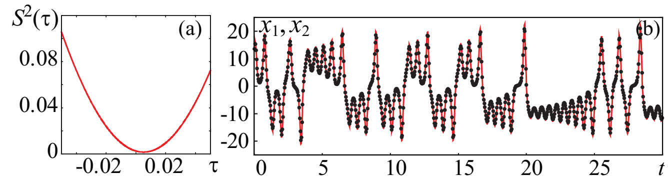

FIG. 4. (Color online) (a) Similarity function $S(\tau)$ and (b) time series of two mutually coupled Lorenz oscillators (5) for $\varepsilon = 7.0$ [see also Figs. 2(a) and 2(b) in [15]]. Since the lag synchronization with the small value of the delay time $\tau \approx 0.007$ is observed in the systems, one time series is shown by the solid line and the second one is displayed by the points.

used in [15], with the same values of control parameters ($\sigma = 10$, $b = 8/3$, and $r_{1,2} = 40{,}35$). Indeed, in Fig. 4 the similarity function (3) and time series of mutually coupled Lorenz systems are shown for the coupling strength $\varepsilon = 7.0$, which corresponds exactly to Figs. 2(a) and 2(b) in [15]. In that work, based on the auxiliary-system approach, Zheng *et al.* have decided that the generalized synchronization does not exist in the system under study. In fact, the coupled Lorenz systems (5) are in the lag synchronization regime (see Fig. 4) and as a consequence in the generalized synchronization regime too.

In conclusion, the concept of partial and global generalized synchronization introduced for chaotic oscillators coupled mutually and complex networks on the basis of the auxiliary-system method extension is misleading. In fact, the auxiliary-system approach may be applied correctly only for the drive-response configuration of networks and coupled oscillators. As far as the generalized synchronization examination in the systems with the mutual type of coupling and networks is concerned, possible ways are the calculation of the spectrum of Lyapunov exponents or the nearest-neighbor method [27] including refinement with the help of the phase tube approach [24].

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