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DYNAMICS CHAOS IN RADIOPHYSICS AND ELECTRONICS

Generalized Synchronization in Networks with a Complicated Topology of Interelement Couplings

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Abstract—A concept of generalized synchronization is proposed. The concept is valid for both two unidirectionally or mutually coupled chaotic oscillators and networks with a complicated topology of interelement couplings and chaotic systems at the nodes. It is shown that the threshold of generalized synchronization formation in systems with 1.5 degrees of freedom can be diagnosed from the instant when the second (positive) Lyapunov exponent enters the domain of negative values. The results obtained are confirmed with the help of the nearest neighbor method. Physical mechanisms of generalized synchronization formation in such systems are revealed.

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INTRODUCTION

One of the known types of the synchronous behavior of coupled chaotic systems is the regime of generalized chaotic synchronization [1–3]. This regime is traditionally considered for a system of two unidirectionally coupled—master and slave—chaotic oscillators $\vec{x}(t)$ and $\vec{u}(t)$, respectively, and means that, after the transient process is completed, unique functional relationship $\mathbf{F}_1[\cdot]$ between the states of these systems is established so that

$$\vec{u}(t) = \mathbf{F}_1[\vec{x}(t)]. \quad (1)$$

Recently, attempts to extend the regime of generalized synchronization to systems with the mutual type of coupling—two mutually coupled chaotic systems [4] and networks with a complicated topology of interelement couplings [5, 6]—have been reported. At the same time, the available studies devoted to this problem are oriented to discovering the existence of this regime only; whereas, as a rule, the concept of generalized synchronization for such systems is not considered. Moreover, in all of the known studies, generalized synchronization is diagnosed with the help of the modified auxiliary system method [7], which is an efficient tool of the analysis of generalized synchronization in systems with the unidirectional coupling. However, the correctness of application of this method to systems with the mutual type of coupling has not yet been proved.

In this study, a universal concept of generalized synchronization is proposed for the first time. This concept is valid for both two unidirectionally or mutually coupled systems and networks of chaotic oscillators with a complicated topology of interelement couplings. It will be shown below that generalized synchronization can be diagnosed in such systems through calculating the Lyuapunov exponents or with the help of the nearest neighbor method. The auxiliary system method applied to systems with the mutual type of coupling yields incorrect results.

1. THE DEFINITION OF GENERALIZED SYNCHRONIZATION AND MECHANISMS OF ITS FORMATION

First, let us extend concept (1) of generalized synchronization to the case of the mutual coupling between systems. In order to take into account the mutual influence of systems, we modify Eq. (1) into the form

$$\mathbf{F}_2[\vec{x}(t), \vec{u}(t)] = 0, \quad (2)$$

for two mutually coupled systems and

$$\mathbf{F}_3[\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_i(t), \ldots, \vec{x}_N(t)] = 0, \quad (3)$$

where $\vec{x}_i(t)$ is the vector of the state of the ith system, for a network of N elements. Equation (2) can be regarded as a particular case of Eq. (3) and holds for both unidirectionally and mutually coupled systems, whereas relationship (1) is a particular case of relationship (2). In other words, we consider generalized synchronizaton of mutually coupled systems to mean the regime such that a unique functional relationship between the states of these systems is realized, but the functional relationship in this case has form (3) rather than (1).

Consider mechanisms of generalized synchronization formation in complex networks with dissipative couplings between elements. It is evident that two unidirectionally or mutually coupled systems can be regarded as the simplest variant of a network consisting of two elements. Therefore, the analysis below is valid for such systems as well.

Assume that there are 3D chaotic dynamic systems[1] at the nodes of a network of N elements. The state of each system is characterized by the state vector $\vec{x}_i = (x_i, y_i, z_i)$, where $i = \overline{1, N}$, and the elements of the network can be nonidentical. In order to reveal mechanisms of generalized synchronization formation in such network, it is convenient to characterize its state with one vector

$$\mathbf{U} = (u_1, u_2, \dots, u_i, \dots, u_{3N})^T,$$

where $u_{3i-2} = x_i$, $u_{3i-1} = y_i$, and $u_{3i} = z_i$. In this case, the evolution of a complex network is determined by the following equation:

$$\dot{\mathbf{U}} = \mathbf{L}(\mathbf{U}) + \varepsilon\tilde{\mathbf{G}}\mathbf{U}, \quad (4)$$

where $\mathbf{L}(\cdot)$ is the vector of evolution of a node element of the network in the absence of coupling, the term $\varepsilon\tilde{\mathbf{G}}$ describes the influence of the network topology and the intensity of interelement couplings. Matrix $\tilde{\mathbf{G}}$, which characterizes the structure of dissipative couplings between elements of the network, is a symmetric matrix such that the sum of elements G_{ij} in each row is 0, $\tilde{G}_{ii} = -\sum_{j \neq i} \tilde{G}_{ij}$ (the dissipativity condition), $\tilde{G}_{ij} = 1$ $(i \neq j)$ when u_i affects u_j and 0 otherwise. Note that all matrix elements $\tilde{G}_{ij}$ are positive or zero with the exception of diagonal elements $\tilde{G}_{ii}$ (which are either negative or sero).

One can easily notice that the term $\varepsilon\tilde{\mathbf{G}}\mathbf{U}$ introduces additional dissipation into system (4). Actually, the dissipation level and the rate of the phase volume contraction in the system under consideration are indicated by the divergence of the vector field

$$\lim_{\Delta t \to 0} \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{\Delta V}{\Delta t} = \operatorname{div}\mathbf{L} + \varepsilon \sum_{i=1}^{3N} \tilde{G}_{ii}, \quad (5)$$

where ΔV is an elementary volume of the phase space of system (4). Since $\tilde{G}_{ii} \leq 0$, the term $\varepsilon\sum_{i=1}^{3N}\tilde{G}_{ii}$ is also negative. Hence, the dissipation in the considered system increases with coupling parameter ε, a circumstance that simplifies the chaotic dynamics of system (4) [8, 9].

In order to characterize the complicacy of chaotic motion, the spectrum of Lyapunov exponents is usually calculated. In the case under study, the behavior of system (4) is described by the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{3N}$ of Lyapunov exponents such that the higher N terms are positive for $\varepsilon = 0$. As dissipation in the system grows, the Lyapunov exponents that were initially positive become negative. Since each Lyapunov exponent characterizes the evolution of a perturbation observed along a certain direction, the transition of one of the Lyapunov exponents from the domain of positive values into the domain of negative values results in the fact that the number of directions along which the phase volume in the $3N$-dimensional phase space of a network of interacting elements is contracted increases by unity. When λ_2 becomes negative, only one positive Lyapunov exponent of the considered network is left, a circumstance that corresponds to the formation of the generalized synchronization regime. It is evident that, in this case, the generalized synchronization regime can be interpreted as switching from hyperchaotic to chaotic oscillations. Note in addition that the negativity of second Lyapunov exponent λ_2, as a criterion for the existence of generalized synchronization, is in agreement with a similar criterion applied for diagnosing generalized synchronization in unidirectionally coupled systems [8, 10].

At the same time, there are studies (see, e.g., [11]) where the transition of one of the positive Lyapunov exponents into the domain of negative values in mutually coupled systems is related with the instant of formation of the time-delay synchronization regime when the interacting systems exhibit identical oscillations shifted by certain time interval τ, i.e., $\vec{u}(t) = \vec{x}(t - \tau)$. However, the difference between the critical values of the coupling parameter that correspond to the transition of a positive Lyapunov exponent into the domain of negative values and to the instant of formation of the time-delay synchronization regime can be rather large. This circumstance is attributed in [11] to the presence of intermittence. Since the time-delay synchronization regime is a particular case of the generalized synchronization regime and occurs at large values of the coupling parameter in the system [12], one can put the following question on the existence of the generalized synchronization regime in mutually coupled systems that differs from the time-delay synchronization regime: Does this regime exist or can the difference of the aforementioned threshold values actually be attributed to the presence of intermittence?

In addition to the calculation of the spectrum of Lyapunov exponents in unidirectionally coupled systems, the auxiliary system method [7] and the nearest neighbor method [1, 13] are widely applied. We should note that the auxiliary system method is most widely spread in practice, because it is easy to realize and because it provides for the high accuracy of the determination of the threshold value for the synchronous regime formation. That is why the auxiliary system method was first generalized to the case of the mutual coupling between systems [4] and networks with a complicated topology of interelement couplings [5, 6].

[1] Note that the analysis below can be extended to systems with an arbitrary dimensionality of the phase space. For the sake of simplicity, we restrict the consideration to a 3D phase space.

According to this method, an element's replica having control parameters identical to those of the element should be considered along with each element of a network. This replica starts with other initial conditions belonging to the same basin of the chaotic attractor and is affected by the same network elements as the original system. These systems should be unidirectionally coupled (see Fig. 1a). Generalized synchronization occurs in this case when all the elements of the network and their replicas start exhibiting in pairs the identical behavior.

At first sight, this generalization of the auxiliary system method to networks of coupled nonlinear elements seems to be rather obvious. At the same time, a more thorough analysis shows that this modification of the method even for two mutually coupled systems can yield incorrect results. In particular, by analogy with study [4], let us consider two mutually coupled systems $\vec{x}(t)$ and $\vec{u}(t)$ having identical control parameters $\vec{x}'(t)$ and $\vec{u}'(t)$, respectively. As has been mentioned in the foregoing, the type of the coupling between the original ($\vec{x}(t)$ and $\vec{u}(t)$) and auxiliary ($\vec{x}'(t)$ and $\vec{u}'(t)$) systems should be unidirectional. Then, according to study [4], when one of the pairs of systems having identical control parameters (e.g., $\vec{x}'(t)$ and $\vec{u}'(t)$) starts exhibiting the identical behavior, the regime of partial generalized synchronization is realized. As soon as the states of the both pairs of identical systems coincide (i.e., $\vec{u}(t) \equiv \vec{u}'(t)$ and $\vec{x}(t) \equiv \vec{x}'(t)$), the regime of complete generalized synchronization is formed in the system.

Now, let us consider the degenerate situation when interacting mutually coupled systems $\vec{x}(t)$ and $\vec{u}(t)$ are identical. Then, the regime of complete chaotic synchronization when $\vec{x}(t) \equiv \vec{u}(t)$ can be realized for the states of interacting systems. Since the auxiliary systems considered according to the method described above are identical, the regimes of partial and complete generalized synchronizations should coincide. Evidently, the generalized synchronization regime realized for the states of mutually coupled systems is in this case equivalent to the regime of complete synchronization between the original and auxiliary systems. Since these are unidirectionally coupled, the complete synchronization regime is realized in such systems later by a time interval that is twice as long as that for the original mutually coupled systems [14]. This means that, according to the terminology of study [4], the generalized synchronization regime is in this case more intense than the complete synchronization regime, which contradicts the definition of the generalized synchronization regime. When the parameters of interacting systems are slightly mismatched, the time-delay synchronization regime rather than complete synchronization is realized. It will be shown below that, in this case, the application of the auxiliary system method for mutually coupled oscillators again

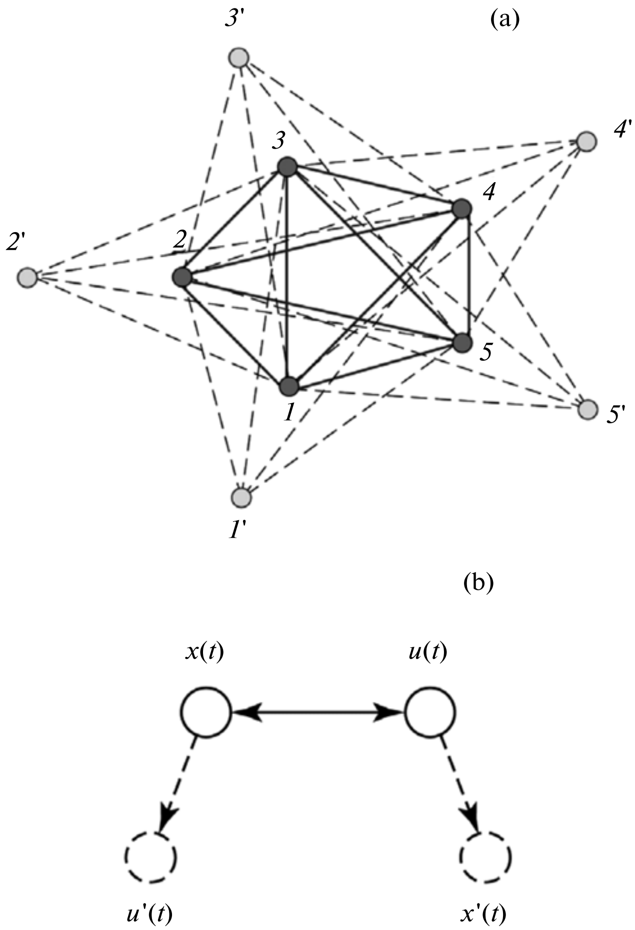

Fig. 1. Illustration of the auxiliary system method for (a) a network of coupled nonlinear elements and (b) two mutually coupled systems.

leads to a contradiction: according to the terminology of study [4], the generalized synchronization regime turns out to be more intense than the time-delay synchronization. It is evident that a similar situation occurs in networks with a complicated topology. Thus, the auxiliary system method proves to be inapplicable for the analysis of generalized synchronization in systems of coupled nonlinear elements.

Another method for diagnosing generalized synchronization in unidirectionally coupled systems is the nearest neighbor method. In this method, the presence of a functional relationship between the states of interacting systems means that all of the close states in the phase space of first system $\vec{x}(t)$ correspond to the close states in the phase space of second system $\vec{u}(t)$ (see [1] for details). For two mutually coupled systems, the converse is valid as well: all of the close states in the phase space of second system $\vec{u}(t)$ correspond to the close states in the phase space of first system $\vec{x}(t)$, and the states of all elements must be close in the case of complex networks. A quantitative characteristic of the degree of closeness of system states is the mean distance between two states of either of systems $\vec{u}^k$ and $\vec{u}^{kn}$ normalized by mean distance δ between randomly chosen states of the other system [13]:

$$d = \frac{1}{N\delta}\sum_{k=0}^{N-1}\left\|\vec{u}^k - \vec{u}^{kn}\right\|, \quad (6)$$

where N is the number of applied iterations. We have $d \to 0$ in the generalized synchronization regime and $d \approx 1$ in the absence of a functional relationship between the states of interacting systems.

However, despite the fact that the nearest neighbor method can easily be applied for the analysis of mutually coupled systems and networks of nonlinear elements, it has a fundamental drawback: this method does not guarantee accurate results. Hence, it enables one to determine the threshold of the synchronous regime formation only approximately and is applied for processing experimental data (when the realization of alternative analysis methods is hampered) or for refining obtained results. Therefore, in this study, we use the nearest neighbor method for verifying the event of generalized synchronization formation in mutually coupled systems and complex networks from the instant when the second (positive) Lyapunov exponent enters the domain of negative values. In order to show that the regime to be diagnosed differs from the time-delay synchronization regime, we diagnose in this study the latter as well.

2. GENERALIZED SYNCHRONIZATION IN TWO MUTUALLY COUPLED SYSTEMS

In order to verify the analysis performed in Section 1, let us consider the behavior of two mutually coupled Rössler systems

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2}, \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \quad (7)$$

where $\vec{x}_{1,2}(t) = (x_{1,2}, y_{1,2}, z_{1,2})^T$ are the state vectors of the interacting systems; ε is the coupling parameter; and $a = 0.15$, $p = 0.2$, and $c = 10$ are the control parameters. Parameter ω_2 characterizing the eigenfrequency of the oscillations of the second system is chosen to be $\omega_2 = 0.95$, while the analogous parameter of the first system was varied within the range [0.89; 1.01] to provide for a mismatch between the interacting oscillators.

It is known that, as coupling intensity ε grows, mutually coupled Rössler systems (7) switch from the asynchronous state to the time-delay synchronization regime [11] through the regime of phase synchronization and synchronization of time scales. As has been noted above, the time-delay synchronization regime is a particular case of the generalized synchronization regime and it is realized at large values of the coupling parameter, i.e., when the time-delay synchronization regime is realized in a system, the generalized synchronization regime is necessarily observed in it as well. At the same time, the diagnostics (proposed in [4]) of the generalized synchronization regime with the help of the auxiliary system method yields incorrect results. In order to confirm this fact, let us compare the threshold values for the formation of these types of synchronous behavior.

Figure 2 shows the boundaries of formation of various types of synchronization in the plane of parameters (ω_1, ε). The boundaries of the partial and complete generalized synchronization regimes diagnosed by means of the auxiliary system method (see Section 1) are also presented in this picture. It is seen that the partial and complete generalized synchronization regimes are generally realized after the point at which the time-delay synchronization occurs. Note that, when the difference between the eigenfrequencies is rather large, these regimes can be formed before the occurrence of the time-delay synchronization.

Thus, the calculated data confirm the theoretical reasoning from Section 1 on the incorrectness of applying the auxiliary system method for the analysis of the behavior of mutually coupled chaotic oscillators. Let us demonstrate that, in this case, the generalized synchronization regime can be diagnosed from the point at which one of the positive Lyapunov exponents enters the domain of negative values.

In addition, Fig. 2 shows the boundary of the passage of the second Lyapunov exponent through the zero value in system (7) (Fig. 2, curve *4*). It is seen that this boundary coincides with none of the critical curves (the boundaries of the time-delay synchronization and the partial and complete generalized synchronizations in the terminology of study [4]) depicted in the figure. Moreover, the aforementioned boundary lies below the boundary of the time-delay synchronization and practically does not depend on the frequency mismatch between the interacting systems.

As we have mentioned in Section 1, in study [1], the transition of one of the positive Lyapunov exponents into the domain of negative values is attributed to the formation of the time-delay synchronization and the difference between the critical values of the coupling parameter that correspond to these regimes is attributed to the intermittent time-delay synchronization [15]. At the same time, as is seen from Fig. 2, the boundaries of formation of both types of the behavior are absolutely independent: the threshold value of the coupling parameter corresponding to the time-delay synchronization formation monotonically grows with the frequency mismatch, while the critical curve characterizing the transition of one of the positive

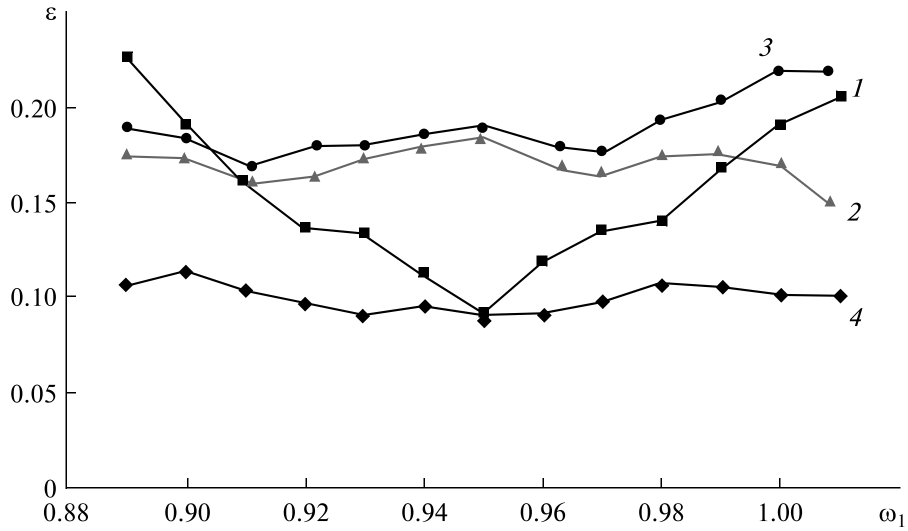

Fig. 2. Boundaries of (*1*) the time-delay synchronization regime and (*2*) partial and (*3*) complete generalized synchronization regimes (in the terminology of study [12]) in system (7) of two mutually coupled Rössler oscillators. Curve *4* corresponds to the point at which one of the positive Lyapunov exponents of system (7) enters the domain of negative values.

Lyapunov exponents into the domain of negative values is practically independent of the mismatch. Moreover, as it has been revealed in the investigations, at various values of the control parameters of the interacting systems, the intermittent behavior can be observed both before and after the point at which a positive Lyapunov exponent passes through the zero value.

Thus, in mutually coupled systems, the transition of one of the positive Lyapunov exponents into the domain of negative values is not related with the time-delay synchronization formation and intermittence near its boundaries. We can suppose that, by analogy with the case of unidirectionally coupled systems, this regime can be interpreted as the transition to the generalized synchronization in mutually coupled systems. To verify the above suppositions, we apply the nearest neighbor method for diagnosing the generalized synchronization in system (7) at $\omega_1 = 0.99$. Quantitative measure d as a function of coupling parameter ε is depicted in Fig. 3b.

In addition, Fig. 3a shows the dependence of the higher four Lyapunov exponents for system (7) at $\omega_1 = 0.99$. The point $\varepsilon_{GS} = 0.106$, at which one of the positive Lyapunov exponent enters the domain of negative values, is marked with the arrows in both of the figures. It is seen that, at this point, quantitative measure d is close to zero, a circumstance that indicates the occurrence of generalized synchronization in system (7).

Let us analyze the character of dependence $d(\varepsilon)$ in more detail. It is seen from Fig. 3b that plane $(\varepsilon; d)$ can symbolically be divided into four regions: region I with $\varepsilon \in [0; 0.04)$, where measure d decreases rather abruptly characterizing switching from the asynchronous state to the phase synchronization regime at $\varepsilon_{PS} = 0.04$; region II with $\varepsilon \in [0.04; 0.09)$, where d is practically constant, a circumstance that indicates the existence of the phase synchronization regime; region III with $\varepsilon \in [0.09; 0.12)$, where measure d decreases very slowly, a circumstance that corresponds to the generalized synchronization formation; and region IV with $\varepsilon > 0.12$, where $d \approx 0$. Note that, in region IV, quantitative value d slightly changes: both before and after the point at which the time-delay synchronization is formed ($\varepsilon_{LS} \approx 0.169$ marked with the arrow in the figure), it remains practically constant. The calculated data also indicate that the partial and complete generalized synchronization regimes (in the terminology of study [4]) do not cause quantitative or qualitative changes in quantitative measure d and in the spectrum of Lyapunov exponents, while all of the remaining types of synchronous behavior are reflected in both of the characteristics.

Phase portraits of interacting Rössler systems are displayed in Figs 3c–3j for various values of coupling parameter ε. In addition, three randomly chosen points $\vec{x}^k$ and their nearest neighbors are shown in each phase portrait of first system $\vec{x}(t)$ (Figs. 3c, 3e, 3g, 3i). Figures 3d, 3f, 3h, and 3j illustrate corresponding states $\vec{u}^{k,kn}$ in the phase space of second system $\vec{u}(t)$.

One can notice that, at small values of the coupling parameter ($\varepsilon = 0.01$), all points in the phase space of the second system are randomly distributed over the entire attractor (Fig. 3 d). As the coupling parameter grows, points start grouping within a finite region of the attractor, and the radius of this region decreases

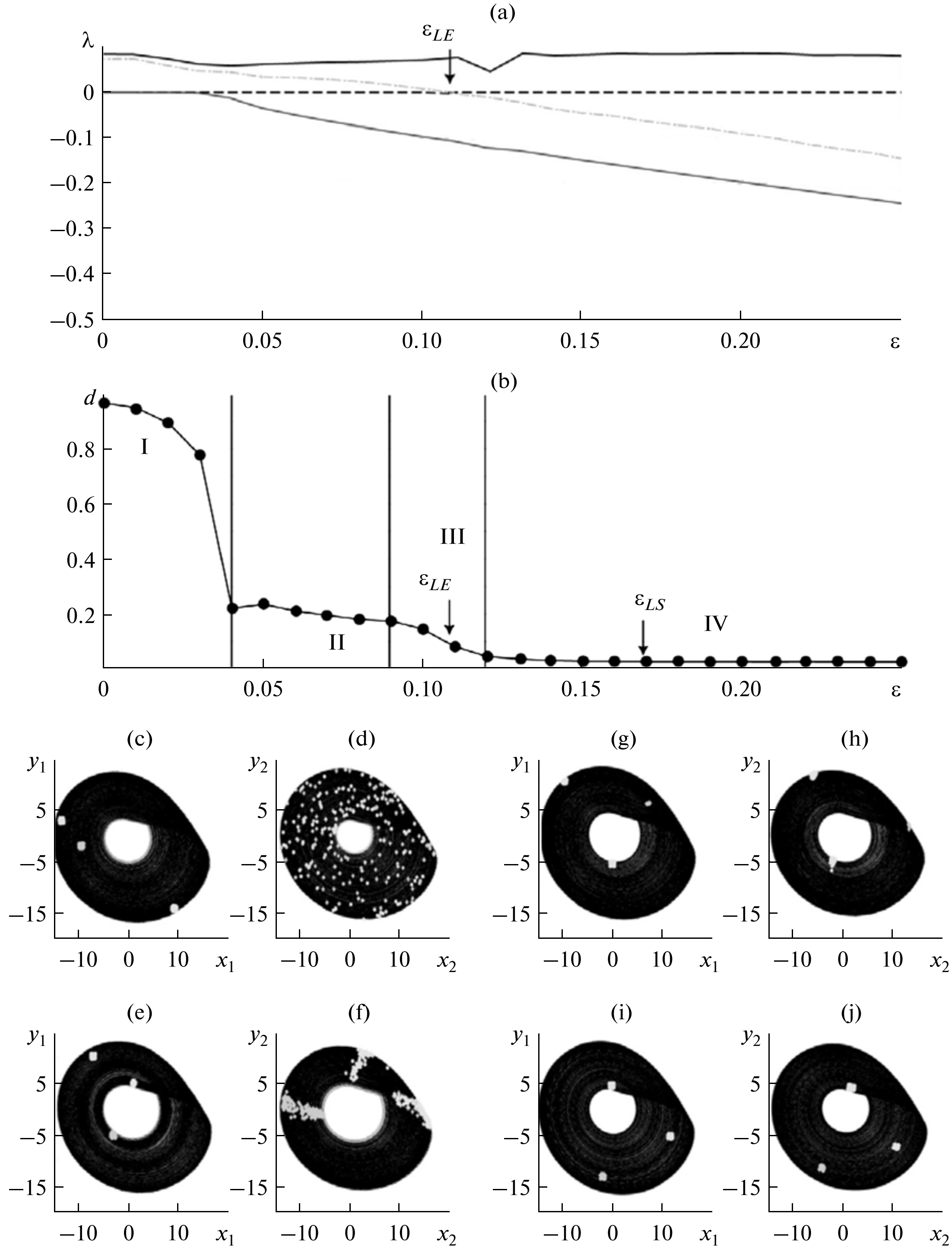

Fig. 3. Dependence of (a) the higher four Lyapunov exponents and (b) quantitative measure d (6) on coupling parameter ε for system (7), $\omega_1 = 0.99$. The critical values of the coupling parameter $\varepsilon_{LE} = 0.106$ (the point at which one of the positive Lyapunov exponents enters the domain of negative values) and $\varepsilon_{LS} = 0.169$ (the point at which the time-delay synchronization occurs) are shown with the arrows. (c–j) The phase portraits of Rössler oscillators for various values of the coupling parameter: (c, d) $\varepsilon = 0.01$ (the asynchronous state), (e, f) $\varepsilon = 0.05$ (the phase synchronization regime), (g, h) $\varepsilon = 0.12$ (the generalized synchronization regime), and (i, j) $\varepsilon = 0.18$ (the time-delay synchronization regime). The chaotic attractors of first system $\vec{x}(t)$ with three randomly chosen points $\vec{x}^k$ and their nearest neighbors $\vec{x}^{kn}$ are shown in Figs. 3c, 3e, 3g, and 3i. Figures 3d, 3f, 3h, and 3j illustrate corresponding states $\vec{u}^{k,kn}$ in the phase space of second system $\vec{u}(t)$.

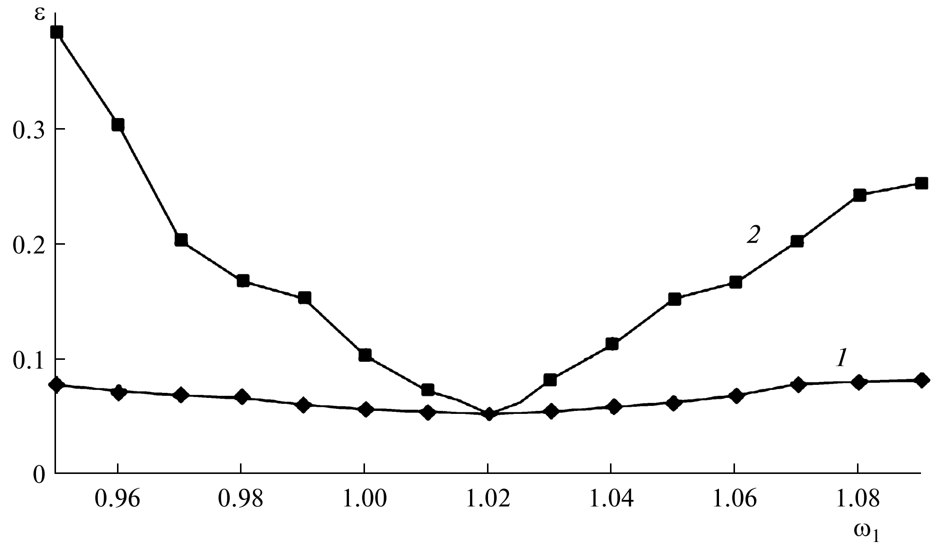

Fig. 4. Boundaries of the (*1*) generalized synchronization and (*2*) time-delay synchronization regimes in system (8) of two mutually coupled tunnel-diode oscillators.

while the coupling parameter increases (cf. Fig. 3f and Fig. 3h). When $\varepsilon > \varepsilon_{LE}$, all the states of the second system that correspond to the nearest neighbors of the first oscillator turn out to be also close and vice versa (Figs. 3g, 3h and Figs. 3i, 3j), a circumstance that proves the generalized synchronization formation. However, we should note a certain difference between the regimes considered: in the time-delay synchronization regime, the representation points corresponding to the nearest neighbors are located practically in the same sections of the chaotic attractor (Figs. 3i, 3j), while these points can be in several different regions in the generalized synchronization regime (Figs. 3g, 3h).

Similar results were obtained for a system of two mutually coupled tunnel-diode oscillators [16]. The equations describing the dynamics of the system have the following form:

$$\begin{aligned}
\dot{x}_1 &= \omega_1^2[h(x_1 - \varepsilon(y_2 - y_1)) + y_1 - z_1], \\
\dot{y}_1 &= -x_1 + \varepsilon(y_2 - y_1), \\
\mu\dot{z}_1 &= x_1 - f(z_1), \\
\dot{x}_2 &= \omega_2^2[h(x_2 - \varepsilon(y_1 - y_2)) + y_2 - z_2], \\
\dot{y}_2 &= -x_2 + \varepsilon(y_1 - y_2), \\
\mu\dot{z}_2 &= x_2 - f(z_2),
\end{aligned} \tag{8}$$

where $h = 0.2$, $\mu = 0.1$, and $\omega_2 = 1.02$ are the control parameters and ε is the coupling parameter. As the dimensionless characteristic of the nonlinear element, the dependence

$$f(\xi) = -\xi + 0.002\sinh(5\xi - 7.5) + 2.9$$

was used.

Figure 4 shows the boundaries of the formation of the generalized synchronization regime (diagnosed from the point at which one of the positive Lyapunov exponent enters the domain of negative values) and the time-delay synchronization regime in system (8). One can easily notice that the behavior of these boundaries is qualitatively similar to the behavior of the corresponding boundaries in a system of two mutually coupled Rössler oscillators (Fig. 2): the threshold of the generalized synchronization is practically independent of the mismatch between the systems, whereas the value of the coupling parameter corresponding to the instant at which the time-delay synchronization regime is formed monotonically grows with the mismatch. Thus, by analogy with mutually coupled Rössler systems, we can apparently speak about the generalized synchronization formation in a system of two mutually coupled tunnel-diode oscillators.

In order to verify this statement, let us consider the behavior of the spectrum of Lyapunov exponents for system (8). Figure 5a shows the dependence of the higher four Lyapunov exponents at $\omega_1 = 1.09$. The point $\varepsilon_{GS} = 0.078$, at which one of the positive Lyapunov exponents enters the domain of negative values, is marked with the arrow.

Figures 5b–5i display the phase portraits of tunnel-diode oscillators (8) for various values of the coupling parameter. As in the case of Rössler systems, three randomly chosen points $\vec{x}^k$ and their nearest neighbors $\vec{x}^{kn}$ are shown in each phase portrait of first system $\vec{x}(t)$ (Figs. 5b, 5d, 5f, 5h) and their images ($\vec{u}^k$ and $\vec{u}^{kn}$, respectively) in the phase space of the second system are presented in Figs. 5c, 5e, 5g, and 5i. It is seen that the character of the disposition of the images of the nearest neighbors in the phase space of the second system

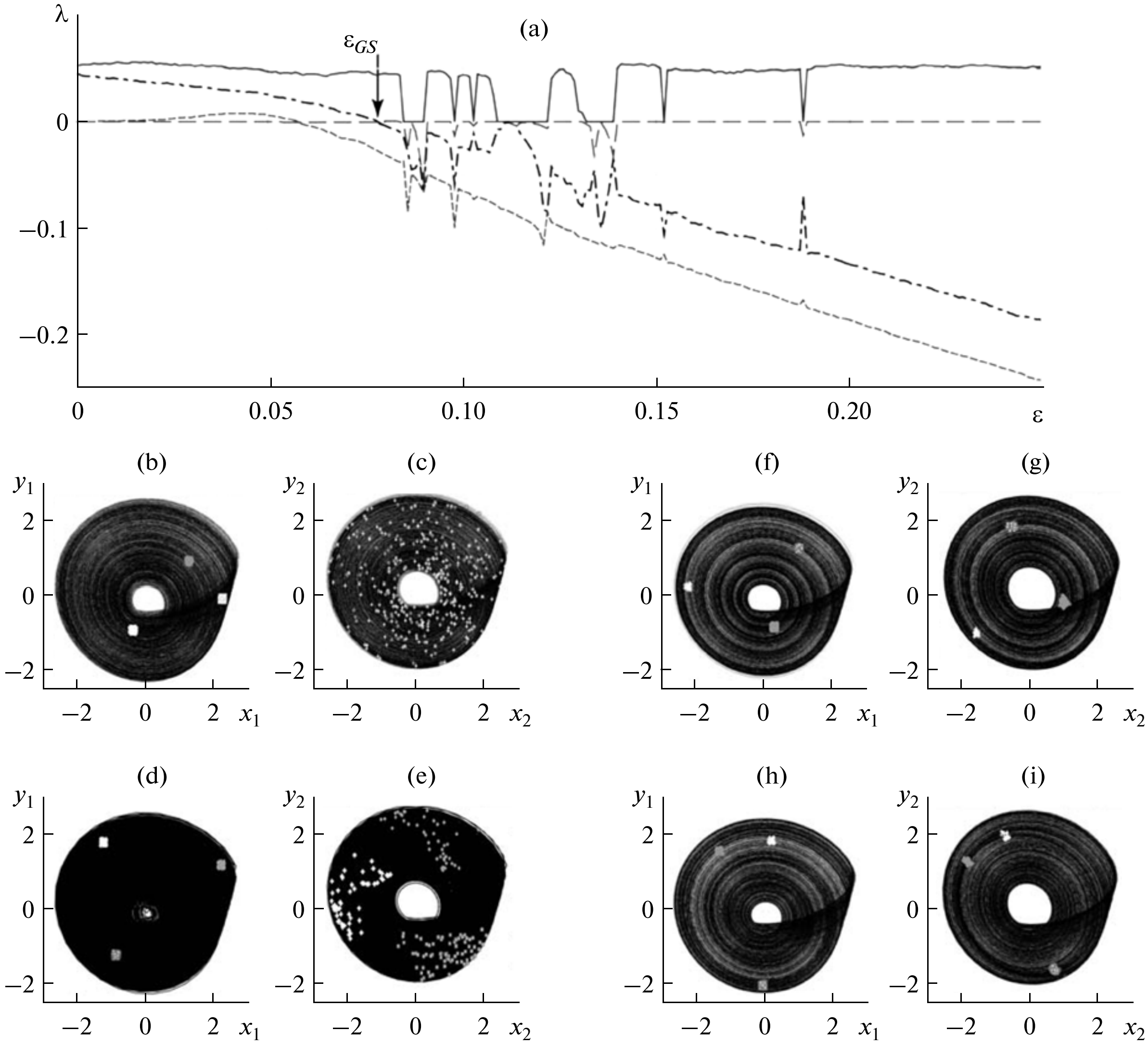

Fig. 5. (a) Dependence of the higher four Lyapunov exponents on coupling parameter ε for system (8), $\omega_1 = 1.09$. The critical value of the coupling parameter $\varepsilon_{GS} = 0.078$ (the point at which one of the positive Lyapunov exponents enters the domain of negative values) is shown with the arrow. The regions of parameters where the maximum Lyapunov exponent vanishes correspond to the periodicity windows in a system of mutually coupled oscillators. (b–i) The phase portraits of tunnel-diode oscillators for various values of the coupling parameter: (b, c) ε = 0.02 (the asynchronous state), (d, e) ε = 0.071 (a regime close to the phase synchronization, (f, g) ε = 0.10 (the generalized synchronization regime), and (h, i) ε = 0.18 (a regime close to the time-delay synchronization). The chaotic attractors of first system $\vec{x}(t)$ with three randomly chosen points $\vec{x}^{k}$ and their nearest neighbors $\vec{x}^{kn}$ are shown in Figs. 5b, 5d, 5f, and 5h. Figures 5c, 5e, 5g, and 5i illustrate corresponding states $\vec{u}^{k,kn}$ in the phase space of second system $\vec{u}(t)$.

resembles that observed in the case of mutually coupled Rössler oscillators (7). In addition, one can distinctly see the difference in the behavior of the representation points corresponding to the nearest neighbors in the generalized synchronization and time-delay synchronization regimes (Figs. 5f, 5g and 5h, 5i).

Thus, we can conclude from the results obtained that, as in the case of unidirectionally coupled chaotic oscillators [12], the formation of generalized synchronization regime in two mutually coupled chaotic systems is due to the reversal of the sign of initially positive Lyapunov exponent λ_2.

3. GENERALIZED SYNCHRONIZATION IN NETWORKS OF COUPLED NONLINEAR ELEMENTS

Now, let us analyze more complicated objects, namely, networks of coupled chaotic oscillators. As a model of such network, we choose a network of $N = 5$ Rössler systems with slightly differing values of param-

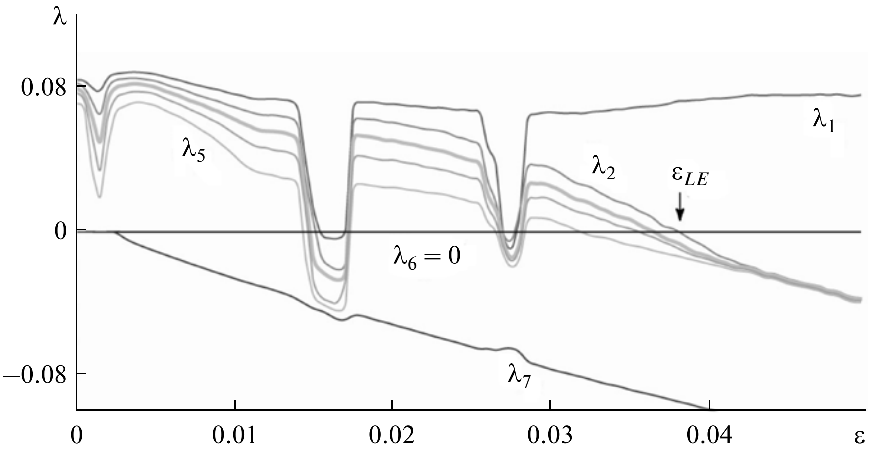

Fig. 6. Dependence of the higher seven Lyapunov exponents on coupling parameter ε for network (9) of five Rössler oscillators. The point $\varepsilon_{LE} = 0.0385$ at which the generalized synchronization regime is realized is shown with the arrow.

eter ω. The evolution of the *i*th element of the network ($i = 1, \ldots, N$) is described by the following system of equations:

$$\begin{aligned}\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon\sum_{j=1}^{N} G_{ij}x_j,\\ \dot{y}_i &= \omega_i x_i + a y_i,\\ \dot{z}_i &= p + z_i(x_i - c),\end{aligned} \quad (9)$$

where the values of control parameters a, p, and c are chosen the same as those used in the case of two mutually coupled oscillators (7), $\omega_1 = 0.95$, $\omega_2 = 9525$, $\omega_3 = 0.955$, $\omega_4 = 9575$, $\omega_5 = 0.96$, $\vec{x}_i(t) = (x_i, y_i, z_i)^T$ is the state vector of the *i*th element, ε is the coupling parameter of the elements, and G_{ij} is an element of coupling matrix **G** of the network. Matrix **G** is similar to that described in Section 1. The topology of couplings between network elements is chosen such that each element of the network is coupled with all of the remaining elements.

The dynamics of a network whose nodes are represented by N Rössler oscillators (9) is characterized by $3N$ Lyapunov exponents. In the absence of couplings between network elements, N exponents are positive, N exponents are negative, and N exponents are zero. As coupling parameter ε grows, the zero and positive Lyapunov exponents gradually enter the domain of negative values, The dependence of the higher seven Lyapunov exponents on coupling parameter ε is depicted in Fig. 6 for a network of five Rössler oscillators.

It is seen that, at $\varepsilon_{LE} \approx 0.0385$, second Lyapunov exponent λ_2 is negative. Hence, for the values $\varepsilon > \varepsilon_{LE}$ of the coupling parameter, the generalized synchronization regime should be formed.

In order to confirm the presence of generalized synchronization in the network under study, we apply the nearest neighbor method by analogy with the case of two mutually coupled systems. The phase portraits of all of the Rössler oscillators of the network are displayed in Fig. 7 for two values of the coupling parameter: one being below (Fig.7a, $\varepsilon = 0.03$) and the other, above (Fig.7b, $\varepsilon = 0.04$) critical value ε_{LE}.

On the phase portraits of three systems $\vec{x}_i(t)$, $i = 2$–4, three points (one for each system) are randomly chosen and their nearest neighbors, i.e., the corresponding points in all of the remaining coupled systems, are found. At $\varepsilon = 0.03$ (Fig. 7a), the points are concentrated in a finite region of the attractor and distributed along the radius, a circumstance that indicates the existence of the phase synchronization regime and the absence of generalized synchronization. When $\varepsilon > \varepsilon_{LE}$ (Fig. 7b), all the states of all systems are close, which indicates the existence of the generalized synchronization regime.

CONCLUSIONS

In this study, the generalized synchronization regime in mutually coupled systems has been investigated. A universal concept of generalized synchronization has been proposed. The concept is valid for both two unidirectionally and mutually coupled chaotic oscillators and complicated networks. It has been shown that the formation of the generalized synchronization regime in interacting 3D dynamic systems is related with the transition of the second Lyapunov exponent into the domain of negative values. Therefore, the generalized synchronization regime can be interpreted as switching from hyperchaotic to chaotic oscillations. The results obtained have been confirmed with the help of the nearest neighbor method. It is important to note that the auxiliary system method, which is widely applied for diagnosing generalized synchronization, yields incorrect results for mutually coupled systems. Since the proposed theory is valid for

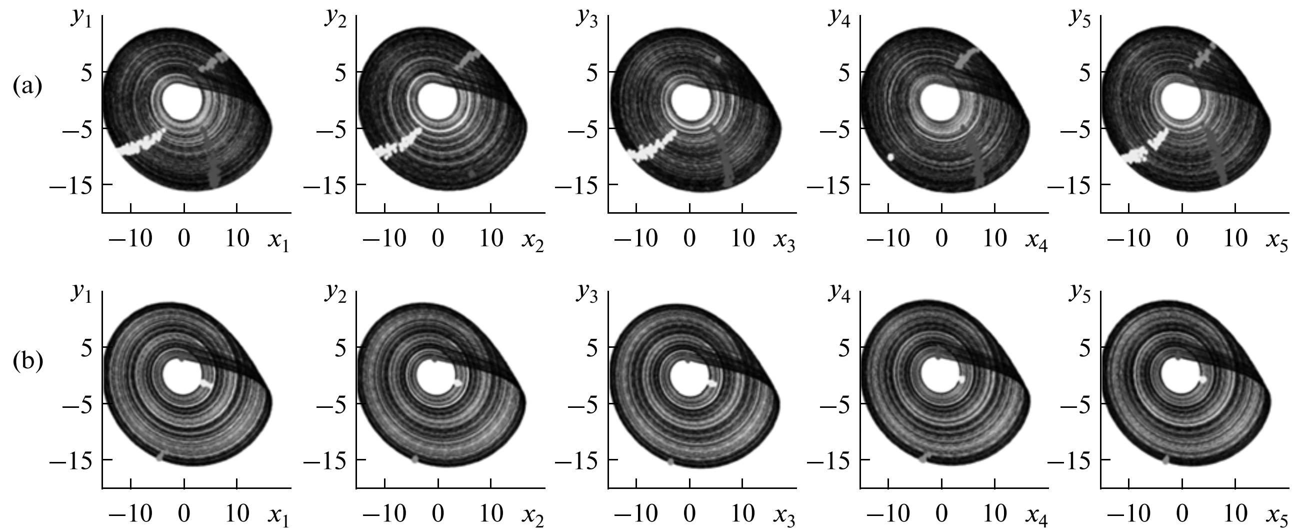

Fig. 7. Phase portraits of five Rössler oscillators for the following two values of the coupling parameter: (a) $\varepsilon = 0.03$ (the phase synchronization regime) and (b) $\varepsilon = 0.04$ (the generalized synchronization regime).

various systems, it can be expected that a similar mechanism is realized in systems of various natures.

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