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Studying the Behavior of Local Lyapunov Exponents near the Boundaries of Synchronous Regime Onset

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Abstract—Local Lyapunov exponents associated with regularities of the transition of the zero and positive conditional Lyapunov exponents in the region of negative values are studied. Local conditional Lyapunov exponents are introduced separately for laminar and turbulent phases. It is shown that the negative nature of these conditional Lyapunov exponents is a manifestation of the synchronism observed at certain time intervals near the boundaries of synchronous regime onset, where synchronous regime has not been established completely.

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Lyapunov exponents are a powerful instrument for analyzing the complicated behavior of systems [1–5]. They are widely used for describing autonomous and nonautonomous dynamics of systems with concentrated and distributed parameters. One of the most important applications of Lyapunov exponents is discovering qualitative changes in the dynamics of a system upon variations in its governing parameters. For example, Lyapunov exponents are used to register the transition from the chaotic regime to hyperchaos [6], to detect the presence of a hyperbolic attractor [3, 7], and for diagnosing different types of chaotic synchronization [8–13].

In addition to Lyapunov exponents, local exponents are sometimes introduced [14, 15] that are calculated for the final time interval. This approach can be used, e.g., in analyzing experimental data when researchers have limited time implementations. Local Lyapunov exponents can also characterize features of the phase space of considered systems or features of their behavior that are manifested at different times. In [16], local Lyapunov exponents were used to explain the behavior of zero Lyapunov exponents near the boundaries of phase chaotic synchronization.

As a rule, the subsequent transition to the region of negative values of two Lyapunov exponents (first zero, then positive) [12, 17] occurs in two bound chaotic systems when the bond parameter changes. As noted above, the change in the sign of the Lyapunov exponent indicates the qualitative changes occuring in the dynamics of the system. In some cases, the transition of one of the Lyapunov exponents into the region of negative values is associated with the emergence of synchronous behavior, as when periodic oscillations are synchronized or the phase and generalized synchronization regimes are established [12, 17]. At the same time, the conditional zero Lyapunov exponent is already quite negative for bound chaotic oscillators at the onset of the phase synchronization regime [12, 18]. The transition of the largest conditional Lyapunov exponent through zero also occurs somewhat earlier than the onset of the generalized chaotic synchronization regime (the difference between critical values of the bond parameter is so small that it is often not taken into account).

Local Lyapunov exponents allow us to explain the processes responsible for the negativity of the zero Lyapunov exponents near boundaries of phase chaotic synchronization [16]. It is known that intermittent behavior is observed below the boundary of the onset of the phase chaotic synchronization regime [19, 20]. The time dependence for the phase difference of a studied system and external impact $\Delta\varphi(t)$ contains regions of synchronous behavior (the laminar phase) interrupted by sudden turbulent eruptions, during which the absolute value of phase difference $|\varphi(t)|$ varies by 2π. The negativity of Lyapunov exponent Λ_0 is associated with the existence of regions with synchronous behavior. To confirm this assumtion, local Lyapunov exponents are considered separately for the laminar and turbulent phases.

As a rule, local Lyapunov exponents are determined for a time interval of fixed length, τ = const. One of the important characteristics of the behavior of a system is the distribution local Lyapunov exponents $N(\lambda_l)$ associated with Lyapunov exponent λ as

$$\lambda = \frac{1}{N_0}\int_{-\infty}^{\infty}\lambda_l N(\lambda_l)d\lambda_l, \quad (1)$$

where $N_0 = \int_{-\infty}^{\infty}N(\lambda_l)d\lambda_l$.

Since our local Lyapunov exponents are used to characterize the dynamics of a system separately for the laminar and turbulent phases, each local Lyapunov exponent is determined for the individual time interval τ corresponding to its phase, laminar or turbulent. Since the durations τ of each phase differ, we must consider distribution $N(\lambda_l, \tau)$ instead of $N(\lambda_l)$. Relation (1) should thus be replaced by

$$\lambda = \frac{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \lambda_l \tau N(\lambda_l, \tau) d\tau}{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \tau N(\lambda_l, \tau) d\tau}. \quad (2)$$

Since we study the local Lyapunov exponents separately for the laminar and turbulent phases, we must consider two distributions: $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ corresponding to the synchronous and asynchronous regions of the system's behavior. In this case, the value of the considered Lyapunov exponent Λ_0 is obviously associated with these distributions as

$$\Lambda_0 = \frac{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \lambda_l \tau \left(N_s(\lambda_l, \tau) + N_a(\lambda_l, \tau) \right) d\tau}{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \tau \left(N_s(\lambda_l, \tau) + N_a(\lambda_l, \tau) \right) d\tau}. \quad (3)$$

It was established earlier [16] that the negativity of the zero Lyapunov exponent Λ_0 in a system of two bound chaotic oscillators near the boundary of the onset of the phase chaotic synchronization regime is associated primarily with the dynamics of the system during the laminar phases. These regions determine the negativity of the Λ_0 value. At the same time, intermittent behavior is also observed near the boundaries of the onset of other synchronous regimes of interacting chaotic systems (e.g., near the boundary of the generalized synchronization regime), and systems with regular dynamics affected by noise. We may thus expect that the mechanisms leading to the transition of the corresponding Lyapunov exponent into the region of negative values discovered in studying the behavior of bound chaotic oscillators near the boundary of phase synchronization are of a universal character and are characteristic for a wide class of systems with chaotic and stochastic dynamics (in addition, there is a close interrelation between deterministic chaos regimes and the regimes observed in systems with regular dynamics affected by noise [11]). To confirm or disprove this supposition, we must consider the behavior of the local Lyapunov exponents near the boundaries of the onset of other synchronous regimes. In this work, we chose the synchronous regime for a nonautonomous Van der Paul generator with noise (a system with periodic dynamics affected by noise) and the regime of generalized chaotic synchronization in discrete maps.

Let us start by considering systems with periodic dynamics affected by noise. It was noted above that a classical Van der Paul self-excited oscillator was chosen as our object of study

$$\ddot{x} - (\lambda - x^2)\dot{x} + x = A\sin(\Omega t) + \xi(t), \quad (4)$$

where $\lambda = 0.1$ is the governing parameter of the system; $\Omega = 0.98$ and A are the frequency and amplitude of the external harmonic impact, respectively; $\xi(t)$ is δ-correlated white noise ($\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(\tau)\rangle = D\delta(t-\tau)$); and D is its intensity. The one-step Euler method with time step $h = 5 \times 10^{-4}$ was used to integrate the system (4).

When there is no noise ($D = 0$), a Van der Paul oscillator exhibits synchronous dynamics at $A > A_c = 0.0238$. The threshold of the emergence of the synchronous regime in this case coincides with the time of the transition of the zero Lyapunov exponent Λ_0 into the region of negative values. Adding noise shifts the threshold value of the synchronous regime onset to higher amplitudes of external impact, and the time of the transition of the zero Lyapunov exponent through zero shifts to lower values of parameter A [12, 21].

Let us study the distribution of the local Lyapunov exponents during synchronous regime onset for the Van der Paul oscillator (4). Benettin's algorithm with Gram–Schmidt orthogonalization was used to calculate the Lyapunov exponents [22].

Figure 1 shows the distribution the local Lyapunov exponents for the laminar and turbulent phases at $D = 1.0$, along with the corresponding lines of the level. The lines of the level of distribution $N_a(\lambda_l, \tau)$ for the local exponents corresponding to the turbulent phases are shown by solid lines, and the analogous lines of the level for the distributions $N_s(\lambda_l, \tau)$ of the local Lyapunov exponents of the laminar phases are shown by the dotted line. The distribution are plotted over $n = 5000$ phases. The value of the supercriticality parameter was set at $A = 0.0245$, which corresponds to the intermittency regime in system (4). It can be seen in the figure that the distribution for the turbulent phases is symmetric with respect to zero, while distribution $N_s(\lambda_l, \tau)$ obtained for the laminar phases is asymmetric and shifts to the region of negative values. The negativity of the considered Lyapunov exponent Λ_0 in system (4) under the impact of external noise is thus associated primarily with the dynamics of the system during the laminar phases. These regions determine the negativity of Λ_0.

Let us consider now the behavior of the largest (conditional) Lyapunov exponent when the bond parameter between systems changes. It was mentioned above that the transition of the largest conditional Lyapunov exponent to the region of negative values indicates the onset of the generalized chaotic synchronization regime in unidirectional bound systems [2, 23]. Let us consider this problem in more detail by

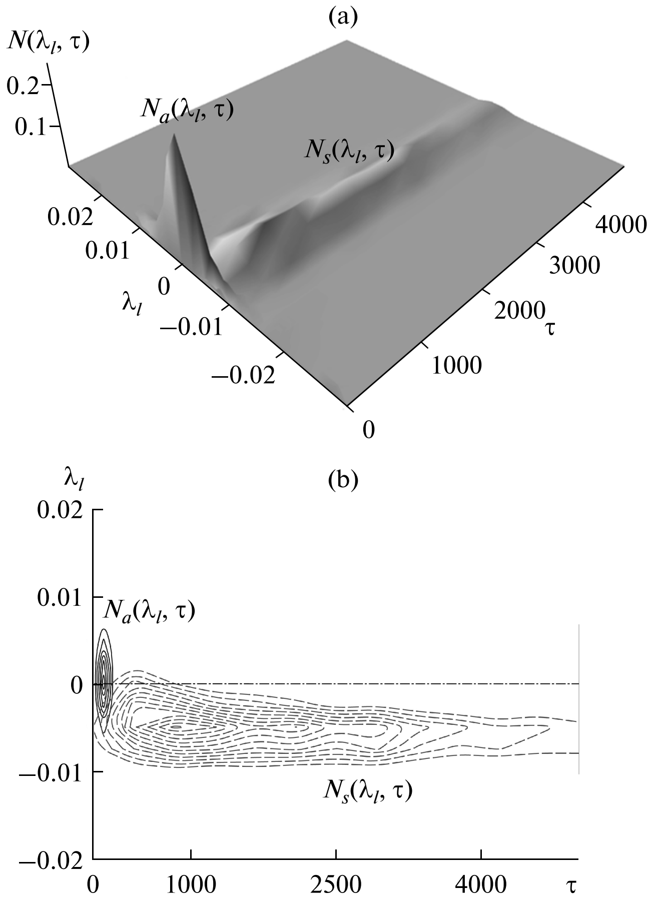

Fig. 1. (a) Normalized distributions of the local Lyapunov exponents obtained for the synchronous and asynchronous phases of chaotic oscillations of Van der Paul oscillator (4) under an external impact in the presence of noise. The distributions are plotted over $n = 5000$ phases. The value of supercriticality parameter $A = 0.0245$; Lyapunov exponent $\Lambda_0 = -0.0042$. (b) Projection of distributions $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ onto plane $((\tau, \lambda_l)$. The lines of the level are shown for turbulent phases with step $h_{N_s} = 2$ (the external (minimum) line of the level corresponds to $N_a = 2$) and for laminar phases with step $h_{N_a} = 2$ (the external (minimum) line of the level corresponds to $N_a = 2$).

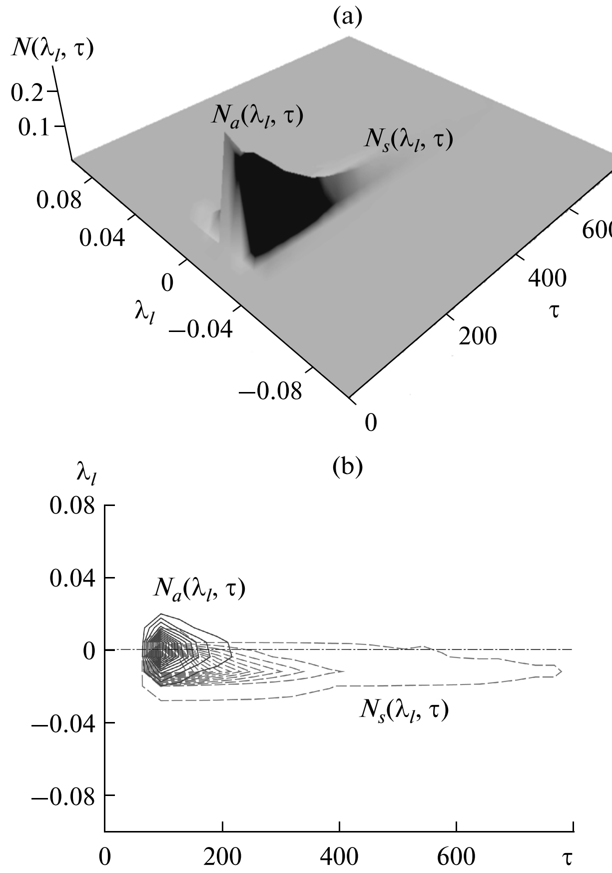

Fig. 2. (a) Distributions of the local Lyapunov exponents obtained for the laminar and turbulent phases of system of unidirectional bound logistic maps (5) plotted over $n = 15000$ phases. Bond parameter $\varepsilon = 0.121$; conditional Lyapunov exponent $\Lambda_r = -0.014$. (b) Projection of distributions $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ onto plane (τ, λ_l).

the example of two unidirectional bound logistic maps:

$$\begin{aligned} x_1^{n+1} &= f(x_1^n, \mu_1), \\ x_2^{n+1} &= f(x_2^n, \mu_2) + \varepsilon(f(x_1^n, \mu_1) - f(x_2^n, \mu_2)), \end{aligned} \qquad (5)$$

where $f(x,\mu) = \mu x(1-x)$, $\mu_1 = 3.75$, $\mu_2 = 3.79$. At these values of the governing parameters, the conditional Lyapunov exponent λ_r is negative at $\varepsilon \in [0.121; 0.174]$ and $\varepsilon \geq 0.294$, indicating that generalized synchronization occurs in these ranges. Weak synchronization occurs in the system in the region of $\varepsilon \in [0.121; 0.174]$; at $\varepsilon \geq 0.294$, the observed regime is strong synchronization [24].

Let us analyze the behavior of local positive Lyapunov exponents at the boundary of generalized synchronization in the studied system. Even though our method for calculating the spectrum of the Lyapunov exponents and that of the auxiliary system diagnose synchronous regime onset at close values of the bond parameter, the transition of the conditional Lyapunov exponent into the region of negative values occurs somewhat earlier than the onset of generalized synchronization. In the region where the conditional Lyapunov exponent is negative but the generalized synchronization regime has not yet begun (this region is very narrow and is often not taken into account), intermittent generalized synchronization takes place, the characteristics of which correspond to intermittency of the on-off type [25]. We may assume that in analogy with the behavior of the zero Lyapunov exponent near the boundary of phase synchronization, the negativity of the largest conditional Lyapunov exponent λ_r is associated with the presence of laminar phases (the region of the synchronous behavior) in the time implementations of interacting unidirectional bound discrete maps (5). To verify this assumption once again, let us consider the local Lyapunov exponents for the laminar and turbulent phases separately. The value of the bond parameter between the interacting systems was set at $\varepsilon = 0.122$.

Figure 2 shows the distributions of local Lyapunov exponents λ_l for the laminar and turbulent phases of system of unidirectional bound logistic maps (5), along with the corresponding lines of the level. In analogy with Van der Paul self-excited oscillator (4), the lines of the level for distribution $N_a(\lambda_l, \tau)$ of the local Lyapunov exponents corresponding to the turbulent phases are shown by solid lines, and the analogous lines of the level for distributions $N_s(\lambda_l, \tau)$ of the local Lyapunov exponents for the laminar phases are shown by the dotted line. All distributions are plotted over $_n =$ 15000 phases. It can be seen in the figure that, as in the case of local zero Lyapunov exponents, the local Lyapunov exponents corresponding to the intervals of synchronous motion are localized in the region of negative values, while the local Lyapunov exponents found for asynchronous regions are located in the vicinity of zero.

CONCLUSIONS

We may conclude that the mechanisms responsible for the negativity of the largest (positive) Lyapunov exponent are the same as for the zero conditional Lyapunov exponent considered above. The negativity of the largest conditional Lyapunov exponent is a consequence of the synchronism observed at definite time intervals corresponding to the laminar phases of system behavior. In addition, there are also turbulent regions in which the largest conditional Lyapunov exponent is positive and close to zero, indicating that the generalized synchronization regime in this case has not yet begun.

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