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Generalized Synchronization in Complex Networks

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Abstract—The phenomenon of generalized synchronization (GS) in networks with a complex topology of links between elements (nodes) representing chaotic dynamical systems has been studied. It is shown that GS onset in these networks can be detected as the moment of transition of the second-order Lyapunov exponent from a positive to a negative value. The results of the analysis are confirmed by the nearest-neighbor method. It is established that the network topology significantly influences the GS development.

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Investigation of the phenomenon of chaotic synchronization of coupled dynamical systems (BPSs) is one of the important areas of research in radio physics and nonlinear dynamics [1]. In simple cases, the chaotic synchronization can be observed between two coupled systems. In recent years, the interest of researchers has focused on studying interactions between a large number of nonlinear systems forming networks [2]. Special attention has been devoted to the analysis of complex networks possessing nonregular structures characterized by considerable nonuniformity of the power of coupling between elements—in particular, dynamical systems exhibiting chaotic behavior [3]. The interest in studying these networks is related both to the need to understand various natural, social, and technogenic phenomena and the importance of fundamental aspects of chaotic synchronization in networks of many coupled partial dynamical systems.

Several types of chaotic synchronization behavior are conventionally distinguished in complex networks, including complete [4, 5], phase [3], generalized [6], cluster synchronization [7], etc. While the complete, phase, and cluster synchronization states in complex networks have been studied in sufficient detail, investigation of generalized synchronization (GS) has just begun. Well-known original works have only been devoted to establishing the existence of GS in a given system, whereas the theory of GS in complex networks has not yet been developed. Moreover, the diagnostics of GS in all cases has been based on modifications of the auxiliary system method [8], which is an effective tool for GS analysis only in systems with unidirectional coupling of components. However, a modification of this method even for two mutually coupled systems leads to incorrect results [9, 10].

This Letter considers the main aspects of the concept of GS in complex networks. It will be shown that GS diagnostics in these systems can be based on calculations of the spectrum of Lyapunov exponents for a given network. It is important to note that the cases of unidirectionally and mutually coupled systems that have been studied previously [11, 12] may be considered partial cases of the proposed concept, which is evidence of its generality.

By analogy with the cases of two unidirectionally or mutually coupled subsystems [9, 13], the GS in a network of coupled nonlinear elements is defined as the regime in which a unique functional relationship,

$$F[x_1(t), x_2(t), \ldots, x_i(t), \ldots, x_N(t)] = 0, \qquad (1)$$

is established between their states, where $x_i(t)$ is the vector of state of the ith element and N is the number of elements in the network. Below, we consider the mechanisms of GS development in complex networks with a dissipative character of coupling between elements (nodes). Let the network nodes represent three-dimensional (3D) dynamical systems exhibiting chaotic dynamics.[1] The state of each system is characterized by the vector $\mathbf{x}_i = (x_i, y_i, z_i)$, where $i = \overline{1, N}$, and the network elements can be nonidentical. In order to elucidate the mechanisms of GS development in the network, its state is conveniently characterized by a single vector $\mathbf{U} = (u_1, u_2, \ldots, u_i, \ldots, u_{3N})^T$, where $u_{3i-2} = x_i$, $u_{3i-1} = y_i$, and $u_{3i} = z_i$. In these terms, the evolution of a complex network is described by the following equation:

$$\dot{\mathbf{U}} = \mathbf{L}(\mathbf{U}) + \varepsilon\tilde{\mathbf{G}}\mathbf{U}, \qquad (2)$$

where $\mathbf{L}(\cdot)$ is the vector of evolution of a nodal element in the absence of coupling, the term $\varepsilon\tilde{\mathbf{G}}$ describes the

[1] It should be noted that the considerations presented below can be generalized to systems with arbitrary dimensionality of the phase space. For simplicity, analysis in this Letter is restricted to a 3D phase space.

influence of the network topology and the degree of coupling between nodes, and $\tilde{\mathbf{G}}$ is a symmetric matrix that characterizes the structure of dissipative links between nodes. The sum of elements $\tilde{G}_{ij}$ in each row of this matrix is zero and the condition of dissipative coupling is

$$\tilde{G}_{ii} = -\sum_{j \neq i} \tilde{G}_{ij},$$

where $\tilde{G}_{ij} = 1$ $(i \neq j)$ if u_i acts upon u_j and $\tilde{G}_{ij} = 0$ otherwise. Thus, all off-diagonal elements $\tilde{G}_{ij}$ are either positive or zero and the diagonal elements $\tilde{G}_{ii}$ are either negative or zero.

It can readily be seen that the term $\varepsilon\tilde{\mathbf{G}}\,\mathbf{U}$ introduces additional dissipation into system (2). Indeed, the level of dissipation and the rate of compression of the phase volume in the system under consideration are characterized by the value of divergence of the vector field:

$$\lim_{\Delta t \to 0} \lim_{\Delta V \to 0} \frac{1}{\Delta V}\frac{\Delta V}{\Delta t} = \mathrm{div}\mathbf{L} + \varepsilon\sum_{i=1}^{3N} \tilde{G}_{ii}, \qquad (3)$$

where ΔV is the elementary volume of the phase space of system (2). Since $\tilde{G}_{ii} \leq 0$, the term $\varepsilon\sum_{i=1}^{3N}\tilde{G}_{ii}$ is also negative. Therefore, an increase in coupling parameter ε leads to growth of dissipation in the network under consideration and simplifies the chaotic dynamics of system (2) [14, 15].

The complexity of chaotic motions is usually characterized by calculating the spectrum of Lyapunov exponents. In the case under consideration, the behavior of system (2) is described by set of Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{3N}$, among which the first N high-order values for $\varepsilon = 0$ must be positive. As the level of dissipation in the system increases, the initially positive Lyapunov exponents gradually become negative. When λ_2 turns out to be negative, the “degree of chaos” decreases to minimum and the system has only one positive Lyapunov exponent, which corresponds to the establishment of a GS regime. Evidently, the GS onset in this case can be interpreted as passage from the hyperchaotic to chaotic oscillations. Note also that use of the negative sign of the second-order Lyapunov exponent, as the criterion of the GS state, agrees with an analogous condition used in the diagnostics of GS regimes between unidirectionally [11, 14] and mutually [9, 12, 16] coupled systems.

For example, let us consider a network consisting of five ($N = 5$) Rössler oscillators with slightly different values of the control parameters. Evolution of the ith element of this system ($i = 1, \ldots, N$) is described by the following system of equations:

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon\sum_{j=1}^{N} G_{ij}x_j, \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= p + z_i(x_i - c), \end{aligned} \qquad (4)$$

where $a = 0.15$, $p = 0.2$, $c = 10$, $\omega_1 = 0.95$, $\omega_2 = 0.9525$, $\omega_3 = 0.955$, $\omega_4 = 0.9575$, and $\omega_5 = 0.96$ are the control parameters; ε is the parameter of coupling between network elements; and G_{ij} are elements of coupling matrix $\mathbf{G}$ of the given network. For simplicity, the topology of links between network elements was selected such that each element was coupled to all other elements.

The dynamics of network (4) is characterized by $3N$ Lyapunov exponents. In the absence of coupling between elements, N of these exponents are positive, N exponents are negative, and the remaining N exponents are zeros. As coupling parameter ε increases, both the zero and positive Lyapunov exponent gradually become negative. Figure 1 presents the dependence of the first seven Lyapunov exponents on coupling parameter ε for the network of five Rössler oscillators. As can be seen, the second Lyapunov exponent becomes negative at $\varepsilon_{\mathrm{LE}} \approx 0.0385$. Therefore, at $\varepsilon > \varepsilon_{\mathrm{LE}}$, the network must exhibit a GS regime.

The obtained results agree well with the nearest-neighbor method [13], according to which the existence of a unique functional relationship between states of the interacting systems implies that all close states in the phase space of one of these systems corresponds to close states in the phase spaces of other systems. Figure 2 shows phase portraits of all Rössler oscillators in the network under consideration for two values of coupling parameter ε, one of which is below (Fig. 2a, $\varepsilon = 0.03$) and the other of which is above (Fig. 2b, $\varepsilon = 0.04$) critical value $\varepsilon_{\mathrm{LE}}$. In the phase portraits of three systems $x_i(t)$ with $i = 2$–4, three points were randomly chosen (one point for each system); then, their nearest neighbors and the corresponding points were determined in all other coupled systems. For $\varepsilon = 0.03$ (Fig. 2a), the points are concentrated in a limited region of the attractor and distributed along the radius, which is evidence of the existence of phase synchronization and the absence of GS. In contrast, for $\varepsilon = 0.04 > \varepsilon_{\mathrm{LE}}$ (Fig. 2b), the states in all systems are close to each other, which is indicative of the GS regime. Thus, the existence of GS in networks of coupled nonlinear elements can be detected as the moment of transition of the second-order Lyapunov exponent from a positive to negative value.

Finally, let us consider the influence of number N of elements and the network topology on the development of GS. Figure 3 shows boundaries of complete synchronization plotted on the N versus ε plane for

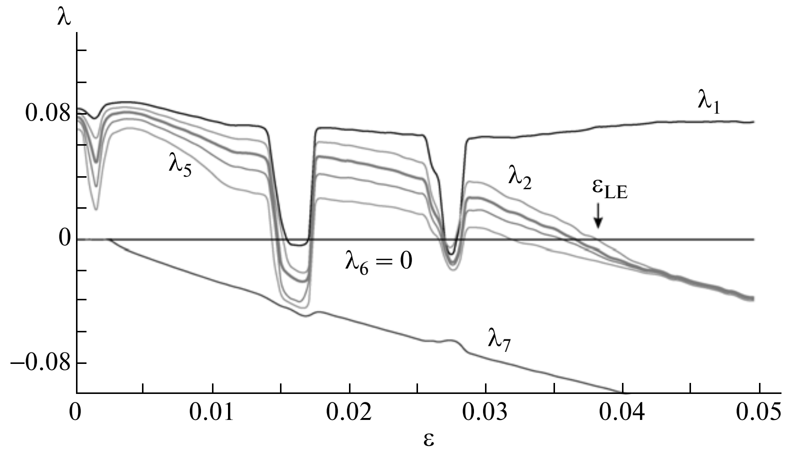

Fig. 1. Plots of the first seven Lyapunov exponents λ vs. coupling parameter ε for a network of five Rössler oscillators described by Eqs. (4). The arrow indicates a critical value of the coupling parameter (ε_{LE}= 0.0385) that corresponds to the onset of GS in the system.

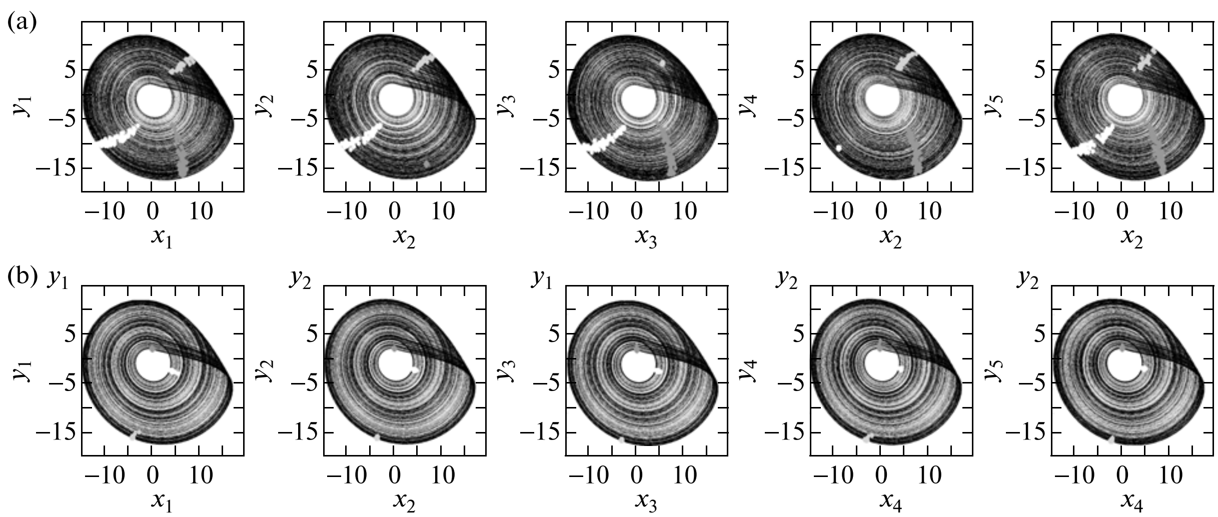

Fig. 2. Phase portraits of five coupled Rössler oscillators for two values of the coupling parameter: (a) ε = 0.03 (phase synchronization regime); (b) ε = 0.04 (GS regime).

networks with various topologies of links between elements, including a random network (curve *1*), regular network (curve *2*), and "small-world" network (curve *3*). The values of control parameters ω_i in all cases were randomly chosen so that the distribution probability density for ω_i obeyed the normal law with a mean value of $\omega_0 = 0.95$ and a variance of $\Delta\omega = 0.017$, which corresponds to a relatively large detuning of the control parameters. As can be seen from Fig. 3, the network topology significantly influences the development of GS. In particular, the threshold of GS onset decreases with increasing ε for the random network, while the analogous thresholds for the regular and small-world networks exhibit monotonic growth. The GS threshold for the regular network increases faster than that for the small-world network. For comparison, Fig. 3 also shows the boundaries of complete synchronization for the same network topologies (dashed curves *1*', *2*', and *3*', respectively). In view of a rather large detuning of the control parameters, the boundaries of GS and complete synchronization in all cases under consideration differ rather significantly, the latter lying significantly higher than the former. However, the qualitative behavior of boundaries (tendency to increase/decrease in threshold values of the coupling parameter, general character of dependences) remains almost unchanged.

Thus, we have studied the phenomenon of GS in networks with a complex topology of links between elements. It is shown that the GS onset in a network of 3D chaotic dynamical systems can be detected as the moment of transition of the second-order Lyapunov

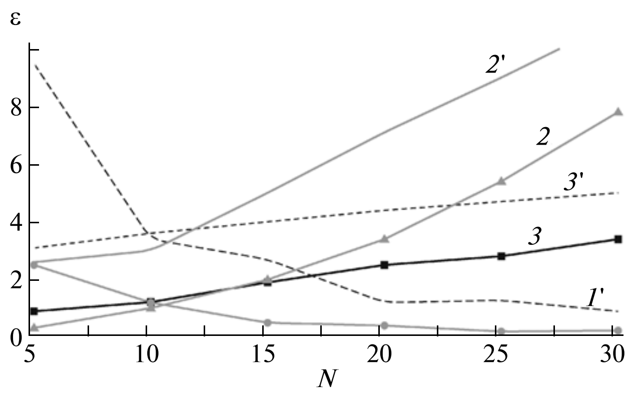

Fig. 3. Boundaries of the GS (solid curves *1–3*) and complete synchronization (dashed curves *1'–3'*) regimes on the *N* vs. ε plane for networks with various topologies of links between elements: (*1*, *1'*) random network, (*2*, *2'*) regular network, and (*3*, *3'*) "small-world" network.

exponent from a positive to negative value. The results of the analysis are confirmed by the nearest-neighbor method. It is established that the number of elements and network topology significantly influence the GS development.

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