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Weak and Strong Generalized Chaotic Synchronization

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Abstract—The existing concept of the weak and strong synchronization in discrete maps is verified. The state vectors of interacting chaotic systems are shown to be related to each other by the functional relation only in the strong synchronization regime; in the weak regime, the prehistory must be taken into account. An approach to determining the threshold of generalized synchronization in such systems is proposed.

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INTRODUCTION

The chaotic synchronization of nonlinear dynamic systems is a universal phenomenon that has important fundamental and practical importance in various branches of science and engineering [1–3]. Synchronization can be observed both in physical and physiological, biological, social, chemical, and other types of synchronous behavior. The most interesting among these is the regime of generalized chaotic synchronization [4–7].

The regime of generalized synchronization is traditionally studied in the system of two unidirectional coupled chaotic oscillators [4, 8–10] and implies that a certain functional relation is established between the states of these systems upon the completion of the transitional period [4]. This functional relation can be quite complicated, and the methods for detecting it are nontrivial. Strong or weak generalized smooth or fractal synchronization can be determined through the type of functional relation [10].The regime of strong synchronization corresponds to a smooth dependence of the coordinates of the drive and response systems (this kind of regime is observed in the case of complete synchronization or a lag synchronization), while in weak generalized synchronization a fractal dependence is observed. In the latter case, two different dynamic systems can act as coupled oscillators, including systems with different dimensions of a phase space, and the diagnostics of the synchronous regime is performed using the auxiliary system approach, as a rule [11].

This work shows the need to verify and improve the existing concept of strong and weak generalized chaotic synchronization of discrete maps. The main reason for this verification is that the very concept of the generalized synchronization of such systems also needs to be improved [12]. It will be shown below that in certain cases, the vectors of states of the interacting chaotic systems in the regime of the generalized synchronization appear to be coupled in such a way that the prehistory of the states of the systems must also be taken into account.

METHOD

Let us consider the behavior of two unidirectionally coupled discrete maps,

$$\begin{aligned} \vec{x}_{n+1} &= \mathbf{H}(\vec{x}_n, \vec{g}_x), \\ \vec{y}_{n+1} &= \mathbf{G}(\vec{y}_n, \vec{g}_y) + \sigma\mathbf{P}(\vec{x}_n, \vec{y}_n), \end{aligned} \tag{1}$$

from the viewpoint of establishing the mode of generalized synchronization. Here, $\vec{x}_n$ [$\vec{y}_n$] are the vectors of the state of the drive [response] systems; **H** and **G** are the evolution operators that determine the vector fields of the systems under study; $\vec{g}_x$ and $\vec{g}_y$ are the vectors of parameters; item **P** is responsible for the unidirectional coupling between the systems; and parameter σ determines the coupling strength between them. We considered a similar situation for flow systems in [12].

As was mentioned above, the regime of generalized chaotic synchronization implies we have the functional relation

$$\vec{y}_n = \mathbf{F}[\vec{x}_n], \tag{2}$$

between the states of drive $\vec{x}_n$ and response $\vec{y}_n$ systems [4, 10]. Further, without loss of generality, we shall assume that the dimensions of the phase space of the interacting systems are identical and equal to m.

It was shown in [12] that in certain cases, the states of the interacting flow systems in the regime of generalized synchronization may be considered as being coupled with each other by a functional, rather than a functional relation. This implies that the state of the response system depends not only on the state of the drive system but also on the prehistory of the behavior of the drive oscillator within a certain time τ. Since the

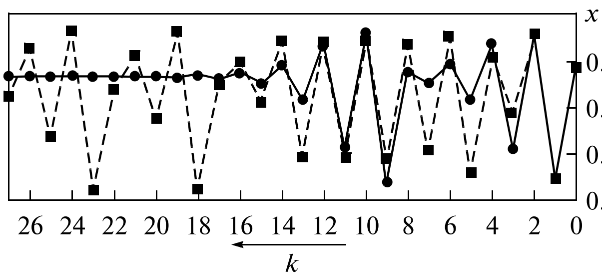

Fig. 1. Dependence of coordinate x of logistic map (14) on the discrete length of prehistory k. Reference trajectory x_{N-k} is shown with a solid line; the trajectories x_{J1-k} and x_{J2-k} (whose final points x_{J1} and x_{J2} are close to x_N) are shown by dots ● (line *1*) and the dashed line with dots ■ (line *2*), respectively.

concept of generalized synchronization (GS) was initially meant for chaotic oscillators regardless of their type, and discrete maps in a number of cases can be obtained from a flow system by plotting the Poincare cross section, similar behavior can be assumed to be observed also for systems with a discrete time. In other words, Eq. (2) for discrete maps must be

$$\vec{y}_n = \mathbf{F}[\vec{x}_n, \vec{x}_{n-1}, \ldots, \vec{x}_{n-K}], \tag{3}$$

where K is the length of discrete prehistory sufficient for a single determination of the state of response map $\vec{y}_n$.

Let $\vec{x}_N$ and $\vec{y}_N$ be the reference trajectories of the chaotic attractors of the drive and response maps that are in the regime of a generalized synchronization, respectively. We also assume that $\delta\vec{y}_{Jk} = \vec{y}_{J-k} - \vec{y}_{N-k}$ and $\delta\vec{x}_{Jk} = \vec{x}_{J-k} - \vec{x}_{N-k}$ ($k = 0, \ldots, K$) are vectors that characterize the deviations of trajectories $\vec{x}_{J-k}$ and $\vec{y}_{J-k}$ from reference states $\vec{x}_{N-k}$ and $\vec{y}_{N-k}$. For the neighboring trajectory $\vec{x}_J$ of the drive map that satisfies the relation $\| \delta\vec{x}_J \| = \| \delta\vec{x}_{J0} \| < \varepsilon$, its image $\vec{y}_J$ in the response system is also close to the reference trajectory $\vec{y}_N$ [4], i.e., $\| \delta\vec{y}_J \| = \| \delta\vec{y}_{J0} \| < \delta(\varepsilon)$.

Assuming that

$$\| \delta\vec{x}_{Jk} \| < \varepsilon, \quad k = 0, \ldots, K, \tag{4}$$

and linearizing (3), we find that

$$\delta\vec{y}_J = \sum_{k=0}^{K} J_{\mathbf{x}_{N-k}} \mathbf{F}[\vec{x}_N, \ldots, \vec{x}_{N-K}] \delta\vec{x}_{Jk}, \tag{5}$$

where $J_{\mathbf{x}_{N-k}}$ is the Jacobian for the kth alternative. Since $\mathbf{F}[\cdot]$ generally cannot be found analytically, Eq. (5) can be rewritten as

$$\delta\vec{y}_J = \sum_{k=0}^{K} \mathbf{A}_k \delta\vec{x}_{Jk}, \tag{6}$$

where $\mathbf{A}_k = J_{\vec{x}_{N-k}} \mathbf{F}[\vec{x}_N, \ldots, \vec{x}_{N-K}]$ ($k = 0, \ldots, K$) is an unknown matrix. It is obvious that the coefficients of matrix $\mathbf{A}_k$ are determined for the entire set of vectors $\vec{x}_{N-K}, \ldots, \vec{x}_N$, but, the elements of this sequence are connected with each other by evolution operator (1), and $\mathbf{A}_k$ may be assumed to depend only on $\vec{x}_{N-K}$, i.e., $\mathbf{A}_k = \mathbf{A}_k(\vec{x}_{N-K})$.

In light of assumption (4) and considering linearity, we can write:

$$\delta\vec{x}_{Jk} = \mathbf{B}_k(\vec{x}_{N-K}) \delta\vec{x}_J, \tag{7}$$

where $\mathbf{B}_k(\vec{x}_{N-K})$ is an unknown matrix[1] whose coefficients depend both on reference vector $\vec{x}_{N-K}$ and on the k number of observed deviation $\delta\vec{x}_{Jk}$, leading to the following expression for $\delta\vec{y}_J$:

$$\delta\vec{y}_J = \sum_{k=0}^{K} \mathbf{A}_k(\vec{x}_{N-K}) \mathbf{B}_k(\vec{x}_{N-K}) \delta\vec{x}_J, \tag{8}$$

and hence

$$\delta\vec{y}_J = \mathbf{C}\delta\vec{x}_J, \tag{9}$$

where $\mathbf{C}$ is a matrix determined as

$$\mathbf{C} = \sum_{k=0}^{K} \mathbf{A}_k(\vec{x}_{N-K}) \mathbf{B}_k(\vec{x}_{N-K}). \tag{10}$$

It is noteworthy that in the scope of the generally accepted concept, generalized synchronization means that the states of interacting systems relate to each other by means of functional relation (2). It is therefore possible to obtain a relation similar to (9),

$$\delta\vec{y}_J = \tilde{\mathbf{C}}\delta\vec{x}_J, \tag{11}$$

the only distinction being that

$$\tilde{\mathbf{C}} = J\mathbf{F}[\vec{x}_n]. \tag{12}$$

Regardless of the visual similarity of (9) and (11), they differ substantially. Indeed, (9) is obtained under the assumption that phase trajectories $\vec{x}_{N-k}$ and $\vec{x}_{J-k}$ ($k = 0, \ldots, K$) are close to each other over the entire length of the prehistory K (see (4)), while (11) demands the proximity of only two points, $\vec{x}_N$ and $\vec{x}_J$. Instead of (4), we must use equation

$$\|\delta\vec{x}_J\| < \varepsilon. \tag{13}$$

Since the phase trajectories for chaotic systems can both agree and disagree, the proximity of $\vec{x}_N$ and $\vec{x}_J$ (Eq. (13)) does not imply the fulfillment of the condition of (4); i.e., only a small number of vectors $\vec{x}_J$ that are close to reference state $\vec{x}_N$ satisfy the condition of (4). This situation is illustrated in Fig. 1 using the example of a logistic map:

$$x_{n+1} = ax_n(1 - x_n), \quad a = 3.75. \tag{14}$$

It can be seen that though both trajectories x_{J1} and x_{J2} are close to reference state x_N (and condition (13) is fulfilled for both trajectories), only trajectory x_{J1} satisfies Eq. (4) owing to the proximity of all trajectories x_{J1-k} (●) to x_{N-k}, while the condition of (4) is not ful-

[1] Except for $\mathbf{B}_0(\vec{x}_{N-K}) = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix.

filled for trajectory x_{J2} (■), since not all points of trajectory x_{J2-k} are close to the points of reference trajectory x_{N-k} over the entire length of prehistory K.

Though all coefficients of matrices $\mathbf{C}$ and $\tilde{\mathbf{C}}$ are unknown, the validity of relation between (9) and (11) can be verified provided that there exists $N > m$ of the nearest neighbors $\vec{x}_{Ji}$ of reference vector $\vec{x}_N$ and their corresponding vectors $\vec{y}_{Ji}$ of the response map. Note also that all vectors $\vec{x}_{Ji}$ that are close to $\vec{x}_N$ must be used to verify Eqs. (11), whereas only vectors $\vec{x}_{Ji}$, whose prehistory satisfies Eq. (4), must be used to verify Eq. (9).

Having determined there is generalized synchronization (e.g., by using the auxiliary system method), m of the nearest neighbors $\vec{x}_{Ji}$ $(i = 1,\ldots,m)$ and their appropriate vectors $\vec{y}_{Ji}$ can also be found to determine the coefficients of matrix $\mathbf{C}$ (or $\tilde{\mathbf{C}}$) using (9) (or (11), respectively) as in [8]. Once the coefficients of matrix $\mathbf{C}$ (or $\tilde{\mathbf{C}}$) are determined, vectors $\delta\vec{z}_{Ji}$ $(i = m+1,\ldots,N)$ can also be found,

$$\delta\vec{z}_{Ji} = \mathbf{C}\delta\vec{x}_{Ji} \quad \text{or} \quad \delta\vec{z}_{Ji} = \tilde{\mathbf{C}}\delta\vec{x}_{Ji} \tag{15}$$

and compared to vectors $\delta\vec{y}_{Ji}$ of the response system to verify the ratio (9) and/or (11).

To characterize the degrees of proximity of vectors $\delta\vec{y}_{Ji}$ and $\delta\vec{z}_{Ji}$ with each other, we must calculate the normalized differences

$$\Delta_{Ji} = \frac{\|\delta\vec{y}_{Ji} - \delta\vec{z}_{Ji}\|}{\|\delta\vec{y}_{Ji}\|} \tag{16}$$

for each pair of vectors and plot their distribution.

To verify the validity of the above theoretical arguments, let us study two unidirectionally coupled logistic maps,

$$\begin{aligned} x_{n+1} &= f(x_n, a_x), \\ y_{n+1} &= f(y_n, a_y) + \sigma(f(x_n, a_x) - f(y_n, a_y)), \end{aligned} \tag{17}$$

where $f(x,a) = ax(1-x)$, $a_x = 3.75$, $a_y = 3.79$ are the control parameters of the drive and response systems, respectively, and σ is the coupling parameter [9]. Since the maps under study are one-dimensional, the vectors must be replaced with scalars, with all theoretical and analytical arguments remaining true.

The threshold of the emergence of a generalized synchronization regime was determined by calculating the conditional Lyapunov exponent for system (17), and was verified using the auxiliary system method [11]. Figure 2 shows the dependence of conditional

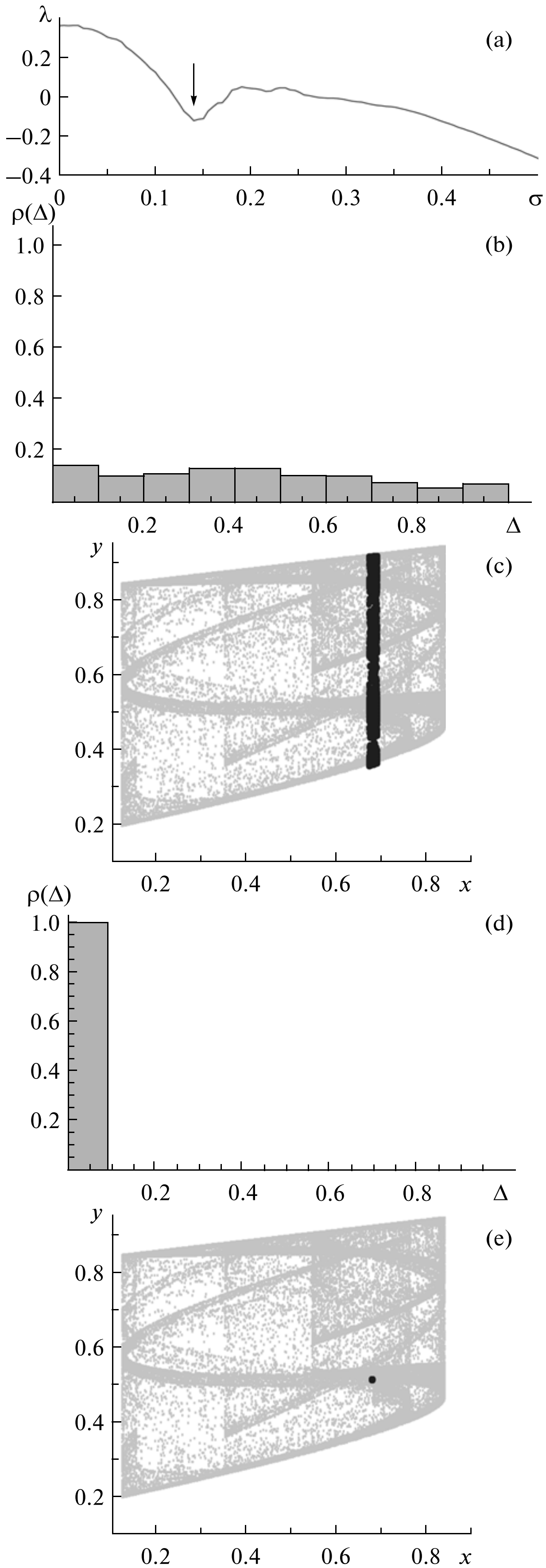

Fig. 2. Dependence of Lyapunov exponent λ on coupling parameter σ (a); histograms of normalized difference Δ_{Ji} (b, d) and plane (x, y) (c, e) for logistic maps (17) in the regime of generalized synchronization (value $\sigma = 0.14$ is designated with an arrow in the figure (a)) for different prehistory lengths: $K = 0$ (b, c), $K = 28$ (d, e). Figures 2c, 2e also show the states of the interacting systems that satisfiy the condition of (4).

Lyapunov index λ on coupling parameter σ. It can be seen that the conditional Lyapunov exponent is negative for $\sigma \in [0.12; 0.18]$ and $\sigma \geq 0.265$, demonstrating that there is generalized synchronization within the specified ranges. Here, generalized synchronization is close to complete (strong) synchronization at rather large parameters of $\sigma \geq 0.265$, for $\sigma \in [0.12; 0.18]$, weak synchronization corresponds to the regime under study. It is obvious that it is unnecessary to consider the prehistory for the regime of a strong synchronization because the states of the interacting systems relate to each other as a simple functional ratio, $y_n \approx x_n$ [10]. At the same time, the case of weak synchronization ($\sigma \in [0.12; 0.18]$) needs additional study.

Below, without loss of generality, we choose a coupling parameter $\sigma = 0.14$ that corresponds to the minimum negative value of the conditional Lyapunov exponent (it is marked by the arrow in Fig. 2a). Assuming that the value of accuracy in (4) is $\varepsilon = 0.0$, let us analyze the influence of the prehistory length K on δz_{Ji} the distribution of normalized difference (16). We arbitrarily select reference point x_N. It is obvious that when the ratio of (9) and (11) is fulfilled, the distribution of normalized differences Δ_{Ji} is a δ function.

Figures 2b, 2d show histograms of the normalized differences Δ_{Ji} at different values of prehistory length K. Figures 2c, 2e show also the planes (x, y) that characterize the states of the drive and response systems for the selected values of the control parameters. It is seen that the states of the interacting systems are coupled with each other by an unsmooth (fractal) ratio, supporting the assumption that the regime being diagnosed is weak. Each of these figures also contains points (x_{Ji}, y_{Ji}) that satisfy the condition of (4). Figures 2b, 2c correspond to a case when all the nearest neighbors are considered (prehistory is ignored, and $K = 0$), i.e., to the traditional concept of generalized synchronization. The values of normalized difference Δ_{Ji} are then evenly distributed over the entire interval [0; 1] (Fig. 2b), and all points in the phase space of the response system are also distributed arbitrarily within a wide range of alternative y (Fig. 2c). The obtained results allow us to infer that Eq. (11) is in this case not fulfilled.

With an increase in prehistory length, the distribution of the normalized differences is transformed. At the optimum prehistory length ($K = 28$), it corresponds to the δ function (Fig. 2d). All states of the system (x_{Ji}, y_{Ji}) that satisfy the condition of (4) then turn out to be concentrated in the narrow vicinity of the reference point (x_N, y_N) (Fig. 2e). Here, fractality disappears and the ratio between the states of the drive and response systems becomes smooth, just as in the case of strong synchronization.

CONCLUSIONS

The concept of generalized synchronization in discrete maps thus needs to be corrected. An approach has been offered for analyzing generalized synchronization in such systems. Our results were shown using the example of two unidirectionally coupled logistic maps in a regime of generalized synchronization.

Our results obtained demonstrate that the division of generalized synchronization into weak and strong also needs revision and verification, since in the regime of strong synchronization, the states of interacting systems appear to be coupled with each other by a functional relation, while in analyzing weak synchronization, the prehistory of the systems' behavior must be considered. Note also that in the cases of both strong and weak synchronization, the ratio between the states of the interacting systems is smooth, and fractality disappears when the prehistory is correctly taken into account.

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