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DYNAMICS CHAOS IN RADIOPHYSICS AND ELECTRONICS

The Influence of the Coupling Mutuality Degree on the Onset of VariousTypes of Chaotic Synchronization

A. A. Koronovskii, O. I. Moskalenko, and S. A. Shurygina

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Abstract—The influence of the degree of coupling mutuality on the onset of the regimes of complete, lag, phase, and generalized synchronization is studied. It is shown that the thresholds of onset of synchronous regimes generally decrease with increasing degree of coupling mutuality between oscillators. It is also found that, for the regime of generalized synchronization in the region of relatively large frequency mismatch, the threshold of onset of synchronous regime weakly depends on the coupling mutuality coefficient.

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INTRODUCTION

The study of synchronization of chaotic oscillations of nonlinear dynamic systems capable of demonstrating a complex behavior has been in the focus of attention of researchers for a long time. Chaotic synchronization is of great fundamental and practical importance (for example, in biological [1, 2], physiological [3, 4], and chemical [5] problems, hidden information transmission using chaotic signals [6, 7], control of microwave electronic systems [8], and so on). A large number of various types of chaotic synchronous behavior are known: complete synchronization [9–15], lag synchronization [16, 17], generalized synchronization [18, 19], phase synchronization [20, 21], and so on. At present, scientists in different countries study these types of synchronization and, while each type of synchronization is rather well studied, the problem of interrelation of different types of synchronous behavior with each other is at the beginning of its development. Thus, for example, papers [16, 20, 22–28] are aimed at finding the interrelation of different types of chaotic synchronization in systems with the same communication type. In [22–26], a new approach to the description of synchronous behavior of chaotic oscillators, the synchronization of time scales, which naturally generalizes different types of synchronous behavior mentioned above is proposed. The application of this approach makes it possible to consider all of the above types of synchronous behavior of chaotic oscillators from a unified point of view.

As a rule, chaotic synchronization is studied either in systems with the unidirectional coupling (when the master oscillator influences the slave one) or with mutual coupling (when both oscillators equally act on each other). Obviously, the system behaviors (in particular, the boundaries of onset of synchronous regimes) differ for different methods of coupling of chaotic systems. It can be expected, for example, that in the case of mutual coupling between chaotic oscillators, the synchronous regime is formed for smaller values of the coupling parameter than in the case of unidirectional coupled systems. Nonetheless, the problem of mutual coupling of different types of synchronous behavior in systems with different coupling types is still open. Therefore, the objective of this study is to determine the regularities of behavior of the boundaries of different types of chaotic synchronization upon transition from unidirectionally coupled systems to systems with mutual coupling.

In this study, we consider four basic types of synchronous behavior: complete synchronization, generalized synchronization, phase synchronization, and lag synchronization. Practically all of these types of chaotic synchronization can take place in systems with unidirectional and mutual coupling types. The problem of transition from one type of coupling to another one is interesting. The regime of generalized chaotic synchronization in mutually coupled systems requires special examination. It should be noted that, in systems with such type of coupling, this regime practically has not been studied. Moreover, the idea of generalized synchronization for mutually coupled systems requires the extension of the existing concept of this type of synchronous behavior in unidirectional coupled systems. Therefore, along with the investigation of mutual coupling of different types of chaotic synchronization in systems with different coupling types, here, we develop the concept of the generalized synchronization valid for unidirectional coupled systems and systems with mutual coupling.

1. TYPES OF CHAOTIC SYNCHRONIZATION

First of all, let us give a brief description of the known types of chaotic synchronization and methods of their diagnostics. The simplest is the regime of complete chaotic synchronization [13, 14], which means the exact matching of state vectors of interacting systems $\vec{x}_1(t) \equiv \vec{x}_2(t)$. Therefore, this regime is possible only in the case of their identical character with respect to control parameters. If the control parameters slightly differ, the regime of lag synchronization [16, 17] can be formed when interacting systems demonstrate identical oscillations shifted by certain time interval τ, i.e., $\vec{x}_1(t) \cong \vec{x}_2(t+\tau)$. Obviously, with increasing the coupling between the oscillators, time shift τ tends to 0, and the synchronous regime tends to the complete chaotic synchronization. The similarity function [16] is used for diagnostics of these types of synchronous behavior,

$$S^2(\tau) = \frac{\left\langle \left|\vec{x}_2(t+\tau) - \vec{x}_1(t)\right|^2 \right\rangle}{\sqrt{\left\langle \left|\vec{x}_1(t)\right|^2 \right\rangle \left\langle \left|\vec{x}_1(t)\right|^2 \right\rangle}}. \quad (1)$$

If the interacting systems are in the regime of lag synchronization, the minimum of the similarity function vanishes, i.e., $\sigma = \min_\tau S(\tau) = 0$, where τ is the time shift between the state vectors of the interacting systems. Obviously, condition $\sigma = 0$ for $\tau = 0$ is the criterion of onset of the regime of complete synchronization. The simplest method of diagnostics of the regime of complete chaotic synchronization is the direct comparison of state vectors of interacting systems $\vec{x}_1(t)$ and $\vec{x}_2(t)$ or the calculation of the synchronization error [27],

$$\langle e \rangle = \int_0^\infty \left\| \vec{x}_1(t) - \vec{x}_2(t) \right\| dt. \quad (2)$$

Generalized synchronization is conventionally considered for a system of two unidirectional coupled chaotic oscillators, master $\vec{x}_d(t)$ and slave $\vec{x}_r(t)$ ones, this synchronization means that, after the end of the transient process, a certain functional dependence between these states is established [18], i.e.,

$$\vec{x}_r(t) = \mathbf{F}\left[\vec{x}_d(t)\right]. \quad (3)$$

The form of this dependence $\mathbf{F}[\cdot]$ may be rather complex, and the procedure of its finding may be quite nontrivial [28]. It should be noted that two different dynamic systems, including systems with different phase space dimensionalities, can serve as the interacting oscillators.

Several methods were proposed for diagnostics of the regime of generalized synchronization between chaotic oscillators, such as the nearest neighbor method [18, 29], the method of calculation of conditional Lyapunov exponents [14, 30], and the auxiliary system method [31].

According to the nearest neighbor method, functional dependence $\mathbf{F}[\cdot]$ between the states of the master and slave systems consists in the fact that two close states in the phase space of the slave oscillator correspond to two close states in the space of the master system [18]. The quantitative characteristic of the degree of closeness of the system states is the mean distance between two states of the slave system, $\vec{x}_r^{\,n}$ and $\vec{x}_r^{\,nn}$ normalized to mean distance δ between the randomly selected states of the master system [32],

$$d = \frac{1}{N\delta} \sum_{n=0}^{N-1} \left\| \vec{x}_r^{\,n} - \vec{x}_r^{\,nn} \right\|, \quad (4)$$

where N is the number of iterations. In the regime of generalized synchronization, $d \to 0$; and, in the absence of a functional relation between the states of the master and slave systems, $d \approx 1$.

It should be noted that the nearest neighbor method for the diagnostics of the regime of generalized synchronization does not make it possible to exactly find the threshold of onset of the synchronous regime; this method can be used to find the range of values of the control parameter to which the point lying between the synchronous and asynchronous regimes belongs. At the same time, the nearest neighbor method does not require the knowledge of the equations describing the system evolution, and therefore, it is used in processing of experimental data [33]. Another application of the nearest neighbor method is the specification and verification of the results obtained using other methods.

For diagnostics of the regime of generalized synchronization based on numerical simulation, the most efficient method is the auxiliary system method [31]. The idea of the auxiliary system method is reduced to the following: along with slave system $\vec{x}_r(t)$, identical auxiliary system $\vec{x}_a(t)$ is considered. The initial conditions the auxiliary system $\vec{x}_a(t_0)$ are chosen different from the initial state of slave system $\vec{x}_r(t_0)$, but in the attraction region of the same attractor (in practice this means a small mismatch of initial conditions, which is automatically implemented due to the presence of fluctuations). In the case of the absence of the regime of generalized synchronization between the interacting systems, state vectors of the slave $\vec{x}_r(t)$ and auxiliary $\vec{x}_a(t)$ systems belong to the same chaotic attractor but are different. If the regime of generalized synchronization takes place, after the end of the transient process, the states of the slave and auxiliary systems should be identical, $\vec{x}_r(t) \equiv \vec{x}_a(t)$, due to the satisfaction of relationships $\vec{x}_r(t) = \mathbf{F}\left[\vec{x}_d(t)\right]$, and correspondingly, $\vec{x}_a(t) = \mathbf{F}\left[\vec{x}_d(t)\right]$. Thus, the equivalence of the states of the slave and auxiliary systems after the transient process is the criterion of the presence of generalized synchronization between the master and slave oscillators.

The analysis of the regime of generalized synchronization can be performed using the calculation of conditional Lyapunov exponents [14, 30]. If the dimensionalities of the phase spaces of the master and slave systems are N_d and N_r, respectively, the behavior of the unidirectional coupled chaotic oscillators can be characterized using the spectrum of Lyapunov factors $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_d+N_r}$. Since the behavior of the master system is independent of the state of the slave oscillator, the spectrum of Lyapunov factors can be separated into two parts: Lyapunov factors of the master system $\lambda_1^d \geq \ldots \geq \lambda_{N_d}^d$ and conditional Lyapunov factors $\lambda_1^r \geq \ldots \geq \lambda_{N_r}^r$. The criterion of the existence of generalized synchronization in unidirectional coupled dynamic systems [14, 28] is the negative character of higher conditional Lyapunov factor λ_1^r. It should also be noted that, for unidirectional chaotic oscillators, the regimes of complete and lag synchronization are also the particular cases of the regime of generalized synchronization [28].

The basis of the concept of chaotic phase synchronization is the idea of instantaneous phase $\varphi(t)$ of chaotic signal [34–37]. Phase synchronization means that the phases of chaotic signals are captured, while the amplitudes of these signals are not coupled with each other and look chaotic [34, 35]. At present, several methods of phase introduction with identically correct results for systems with a rather good topology of the attractor (systems with phase coherent attractor) are known. The most widely spread is the introduction of phase $\varphi(t)$ as the angle in the polar coordinate system in plane (x, y) [16, 38],

$$\varphi = \arctan\frac{y}{x}, \tag{5}$$

but, in this case, all trajectories in plane (x, y) should rotate about the point of origin. Sometimes, it is possible to transform coordinates in order to obtain a projection suitable for the phase introduction (as, for example, for the Lorentz system) [20, 38]. In a number of cases, the transition to velocity plane $(\dot{x}, \dot{y})$ makes it possible to eliminate the phase incoherence of the attractor and introduce the phase as the angle in polar coordinates in plane $(\dot{x}, \dot{y})$, i. e., $\varphi = \arctan\frac{\dot{y}}{\dot{x}}$ [39].

Phase synchronization occurs if difference of instantaneous phases of chaotic signals $\vec{x}_{1,2}(t)$ is time-limited,

$$|\varphi_1(t) - \varphi_2(t)| < \text{const}. \tag{6}$$

2. INFLUENCE OF THE DEGREE OF COUPLING MUTUALITY ON THE ONSET OF REGIMES OF CHAOTIC SYNCHRONIZATION: ANALYTICAL ESTIMATES

Let us study the influence of the type of coupling between the coupled systems on the onset of the regimes of complete, lag, generalized, and phase synchronization.

Let us consider the behavior of two dissipatively coupled identical chaotic oscillators,

$$\begin{aligned} \dot{\vec{x}}_1 &= \mathbf{F}(\vec{x}_1) + \sigma\alpha\mathbf{H}(\vec{x}_2 - \vec{x}_1), \\ \dot{\vec{x}}_2 &= \mathbf{F}(\vec{x}_2) + \sigma\mathbf{H}(\vec{x}_1 - \vec{x}_2), \end{aligned} \tag{7}$$

where $\vec{x}_i$ is the state vector of ith element, $\mathbf{F}$ is the operator of the system evolution, $\mathbf{H} = \delta_{i,j}$ is the matrix characterizing the coupling between the elements, σ is the scalar parameter characterizing the coupling intensity, $\delta_{i,j} = 0$ or 1, $\delta_{i,j} = 0$ $(i \neq j)$, α is the coupling mutuality coefficient characterizing the degree of influence of the second system on the first one (or the symmetry of coupling between the systems). It is clear that, for $\alpha = 0$, the type of coupling in system (7) is unidirectional, and $\alpha = 1$, on the contrary, characterizes the mutual symmetric coupling between the interacting systems.

Due to the identical character of the values of control parameters of interacting systems, the regime of complete chaotic synchronization can be formed in system (7). It is possible to find analytically the threshold value of the onset of the regime of complete chaotic synchronization using the method of determination of the stability of the synchronous state of the network based on the examination of the higher Lyapunov factor [40, 41]. According to this method, for determination of the threshold of the onset of complete chaotic synchronization (and the stability domain of this regime) in an arbitrary network consisting of N identical coupled oscillators,

$$\dot{\vec{x}}_i = \mathbf{F}(\vec{x}_i) + \sigma\sum_{j=1}^{N}\mathbf{G}_{ij}\mathbf{L}(\vec{x}_j), \quad i = 1, 2, \ldots, N \tag{8}$$

(where $\mathbf{L}$ is the operator determining the mutual coupling of elements, $\mathbf{G}_{ij}$ are the elements of coupling matrix $\mathbf{G}$), it is sufficient to calculate the dependence of higher Lyapunov factor Λ on parameter $\nu = \sigma\lambda_i$ (λ_i, $i = 2, \ldots, N$ are the eigenvalues of coupling matrix $\mathbf{G}$) for the system

$$\dot{\xi} = [\mathbf{JF}(\vec{x}_s) + \nu\mathbf{JL}(\vec{x}_s)]\xi, \tag{9}$$

where $\vec{x}$ is the variable characterizing the time evolution of the autonomous oscillator,

$$\dot{\vec{x}} = \mathbf{F}(\vec{x}). \tag{10}$$

The domains in which $\Lambda(\nu) < 0$ correspond to the stability domains of the synchronous state of the network. In this case, instant of transition ν_1 of the higher Lyapunov factor over zero, $\Lambda(\nu_1) = 0$, can be considered as the threshold of the onset of the regime of complete chaotic synchronization in the network of coupled nonlinear elements [43].

Obviously, for $i = 2$, system (8) is becomes system (7), i.e., two coupled systems (7) represent the simplest case of a network, a network of two elements. In this case, the coupling matrix between the network elements is

$$\mathbf{G}_{ij} = \begin{pmatrix} -1 & 1 \\ \alpha & -\alpha \end{pmatrix}, \tag{11}$$

and its eigenvalues $\lambda_1^\alpha = 0, \lambda_2^\alpha = -(1+\alpha)$. The threshold of onset of complete synchronization in this case is completely determined by the value of parameter α, i.e., the degree of influence of the second system on the first one (the degree of coupling mutuality). The threshold of onset of complete synchronization is

$$\sigma^\alpha = \frac{\nu_1}{|\lambda_2^\alpha|} = \frac{\nu_1}{1+\alpha}. \tag{12}$$

It follows from relationship (12) that the thresholds of establishment of complete synchronization in systems with unidirectional $\sigma^1(\alpha = 0)$ and mutual $\sigma^2(\alpha = 1)$ types of coupling are related as

$$\sigma^1/\sigma^2 = 2/1. \tag{13}$$

It should be noted that relationships (12) and (13) should be satisfied for any dynamic systems exhibiting chaotic dynamics that serve as the elements of the network with the dimensionality 2. At the same time, this relationship is valid only for the regime of complete chaotic synchronization, whose implementation is possible in systems with identical parameters. It is clear that the introduction of mismatch of the control parameters of the systems results in the violation of relationships (12), (13) and implementation of other types of synchronous behavior in system (7). In this case, it is quite difficult to find analytically the relationships of the threshold values of onset of synchronous regimes. At the same time, numerical simulation makes it possible to obtain corresponding estimates for different types of synchronous behavior in coupled systems with identical and detuned parameters. The next section gives such estimates for different types of chaotic synchronization, as exemplified by coupled Rossler systems.

3. INFLUENCE OF THE DEGREE OF COUPLING MUTUALITY ON THE ONSET OF REGIMES OF CHAOTIC SYNCHRONIZATION: NUMERICAL SIMULATION

The equations describing the dynamics of the studied system of coupled Rossler oscillators are written as

$$\begin{gathered} \dot{x}_1 = -\omega_1 y_1 - z_1 + \alpha\sigma(x_2 - x_1), \\ \dot{y}_1 = \omega_1 x_1 + a y_1, \quad \dot{z}_1 = p + z_1(x_1 - c), \\ \dot{x}_2 = -\omega_2 y_2 - z_2 + \sigma(x_1 - x_2), \\ \dot{y}_2 = \omega_2 x_2 + a y_2, \quad \dot{z}_2 = p + z_2(x_2 - c), \end{gathered} \tag{14}$$

where $(x_{1,2}, y_{1,2}, z_{1,2})$ are the Cartesian coordinates of the first and second systems, respectively; overdots denote time derivatives, σ are the coupling parameters, and α is the coupling mutuality coefficient. The values of the other control parameters of system (14), are chosen by analogy with [44, 45] as follows: $a = 0.15$, $p = 0.2$, and $c = 10.0$. Parameter ω_2 (characterizing the eigenfrequency of oscillations of the second system) is taken equal to $\omega_2 = 0.95$, and the similar parameter of the first system varies in the range [0.89, 1.01] in order to determine the mismatch between the oscillators.

Let us study the behavior of the boundaries of onset of the regimes of complete, lag, generalized, and phase synchronizations under variation of parameter $\alpha \in [0,1]$.

A. COMPLETE SYNCHRONIZATION

First, let us consider the relationship between the analytical estimates and numerical results for the regime of complete chaotic synchronization. In this case, system of equations (14) can be written in form (8), where $\vec{x}_i = (x_i, y_i, z_i)$, $\mathbf{F}(\vec{x}_i) = (-\omega_i y_i - z_i, \omega_i x_i + a y_i, p + z_i(x_i - c))$, $\mathbf{H}(\vec{x}_i) = (x_i, 0, 0)^T$, $i = 1, 2$, $\omega_i = 0.95$, and the matrix of the coupling coefficients between the elements of the network is determined by relationship (11). Figure 1 shows the dependence of the higher Lyapunov factor for this network on parameter ν. The value of $\nu_1 = 0.2$, corresponding to the instant of transition of the higher Lyapunov factor over zero, is marked by the arrow.

The following conclusion can be made based on the calculation: the dependence of the threshold of onset of the regime of complete chaotic synchronization in the system of two coupled Rossler oscillators (14) is determined by relationship (12), where $\nu_1 = 0.2$. Let us compare thus obtained analytical estimates with the results of direct numerical calculations. Figure 2 shows the dependence of the threshold of onset of complete chaotic synchronization on parameter α obtained by direct comparison of the state vectors of the interacting systems (see Section 1). This figure also shows theoretical dependence (12). Good agreement of the analytical and numerical results can be seen. Thus, with increasing coefficient α, the threshold of onset of complete synchronization decreases according to (12). In this case, it can be clearly seen that the regime of complete synchronization in unidirectional coupled identical Rossler systems (14) occurs for coupling intensity $\sigma^1 = 0.2$, while, for identical mutually coupled Rossler systems, it is implemented for $\sigma^2 = 0.1$; i.e., the thresholds of onset of complete synchronization in unidirectional and mutually coupled Rossler systems are related as 2 : 1, in complete agreement with the results of theoretical predictions.

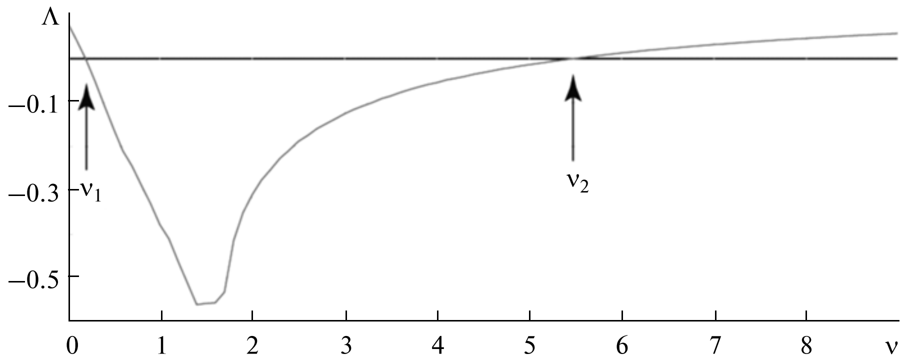

Fig. 1. Higher Lyapunov factor as a function of parameter ν for a network of Rossler systems. Boundaries $\nu_{1,2}$ of stability of synchronous state of the network are shown by arrows.

B. LAG SYNCHRONIZATION

Now, let us study the problem of the influence of mismatch of the control parameters on onset of the synchronous regime in coupled systems. It is known that, in this case, lag synchronization can be established when the interacting systems demonstrate identical oscillations shifted by a certain time interval. It has been noted above that the analytical calculation of the boundary of lag synchronization (as well as other types of synchronous behavior considered below) is difficult. However, it can be easily found numerically by calculating likelihood function (1).

Let us consider the behavior of the boundaries of the regime of lag synchronization under variation of parameter α. Figure 3a shows these boundaries on parameter plane (ω_1, σ) for various $\alpha \in [0;1]$. Curves *1* ($\alpha = 0$) and *6* ($\alpha = 1$) show the boundaries of lag synchronization for unidirectional and mutually coupled Rossler oscillators, respectively. The other curves are between them, and the larger α, the lower the curve, and the faster the regime of lag synchronization is realized. At the same time, it is impossible to observe strict law $\sigma(\alpha)$. Figure 3b shows the dependences $\sigma(\alpha)$ for the case of relatively large and relatively small eigenfrequency mismatch.

The dependences illustrating the value of the time shift between the states of the interacting systems at the instant of onset of the regime of lag synchronization for various values of parameter ω_1 for unidirectional and mutually coupled oscillators are shown in Fig. 4. It can be easily seen that, if the control parameters of the interacting systems are identical, the time shift between the state of the interacting systems in the case of unidirectional and mutual coupling is equal to zero, and the regime of lag synchronization coincides with the regime of complete synchronization.

As the mismatch between the systems increases, the time shift between the system states grows. If the time shift becomes relatively small (which corresponds to the case of a relatively small mismatch of control parameters, $\omega_1 \in (0.93; 0.97)$), relationship (13) is still valid, although approximately. As the frequency mismatch increases, the time shift practically reaches saturation (see Fig. 4 for $\omega_1 < 0.93$ and $\omega_1 > 0.97$), and relationship (13) is violated (see Fig. 3a).

Note that the time shift reaches saturation in systems with unidirectional coupling earlier than in similar mutually coupled systems. Moreover, the time shift between the states of the interacting systems in the case of mutual coupling between them is larger than in systems with unidirectional coupling. This is determined by substantial differences in the threshold values of the coupling parameter corresponding to the onset of the synchronous regime. Since the time shift between the states of the interacting systems in the regime of lag synchronization depends on the coupling parameter as $\tau \sim \sigma^{-1}$ [23] and the growth of the mismatch between the systems results in the sharp growth of the threshold value of the coupling parameter, this behavior of the dependences shown in Fig. 4

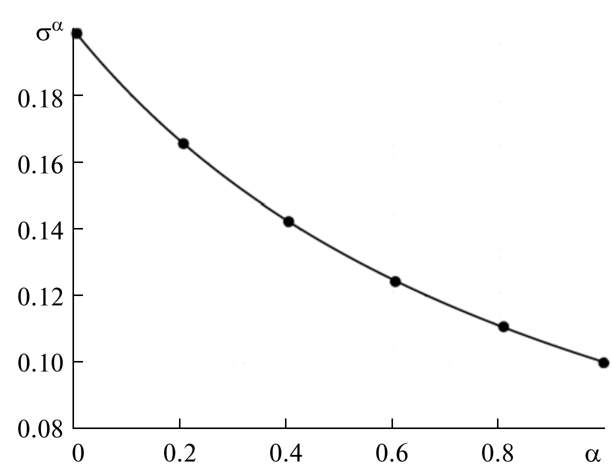

Fig. 2. Threshold of onset of complete synchronization as a function of coupling mutuality coefficient α in system (14) ($\omega_{1,2} = 0.95$): (dots) numerical simulation and (solid line) calculation by formula (12).

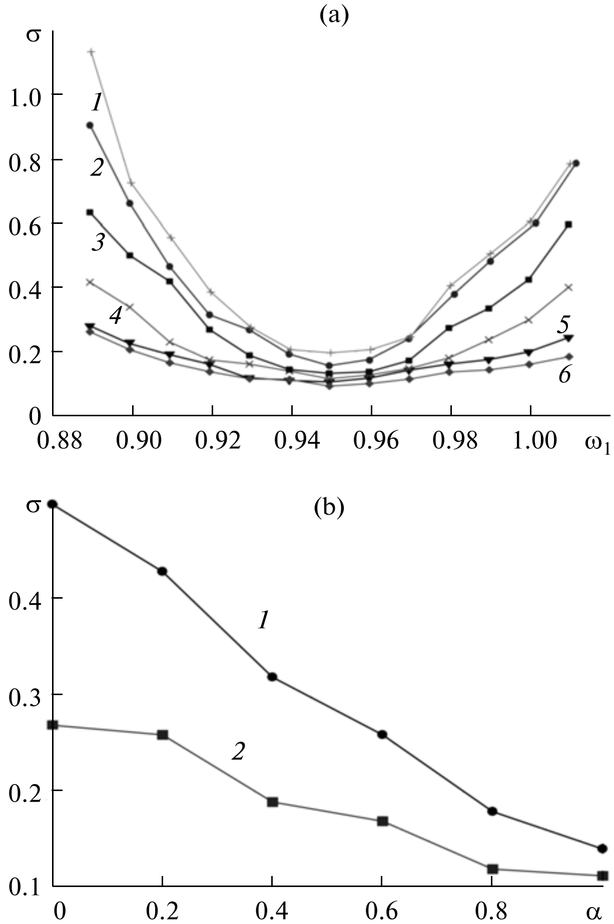

Fig. 3. (a) Boundaries of lag synchronization of two coupled Rossler systems (14) in parameter plane (ω_1, σ) with α increasing from (curve *1*) 0 to (curve *6*) 1 with a step of 0.2; (b) lag synchronization threshold as a function of α for relatively large ($\omega_1 = 0.99$, curve *1*) and relatively small ($\omega_1 = 0.93$, curve *2*) eigenfrequency mismatches.

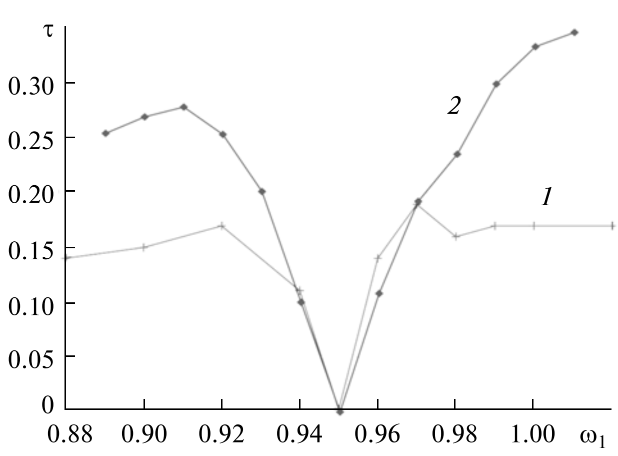

Fig. 4. Time shift between the states of two (*1*) unidirectional and (*2*) mutually coupled Rossler systems at the instant of establishment of lag synchronization for various values of parameter ω_1.

turns out to be rather typical of unidirectionally and mutually coupled chaotic systems.

C. GENERALIZED SYNCHRONIZATION

Now, let us analyze generalized synchronization [18] in system (14). It has already been mentioned above that, for unidirectional coupled systems, the regimes of complete and lag synchronizations are particular cases and stronger forms of generalized synchronization (see also [28]); i.e., if the regime of complete or lag synchronization is implemented in the system, generalized synchronization should necessarily be realized. A similar situation should take place in systems with mutual coupling, both symmetric and nonsymmetric; however, thereare a number of serious questions requiring further study. Recall that the idea of generalized synchronization for systems with mutual coupling has not been introduced. Therefore, similarly to the case of unidirectional coupled systems, we assume that generalized synchronization in two systems with mutual coupling is the regime for which the unique functional relationship between their states is established. In this case, functional relationship (3) is rewritten as

$$\mathbf{F}[\vec{x}_1(t), \vec{x}_2(t)] = 0. \tag{15}$$

It should be noted that relationship (3) can be considered as a particular case of (15); therefore, the main properties of generalized synchronization of unidirectional and mutually coupled systems should be retained.

It has already been mentioned above that, for diagnostics of generalized synchronization in systems with unidirectional coupling, along with the auxiliary system method and the nearest neighborhood method, the method of calculation of Lyapunov exponents is used; the latter can be applied for the analysis of generalized synchronization in mutually coupled systems. Indeed, if the coupling parameter between the systems increases, one of the positive Lyapunov exponents goes over to the domain of negative values (Fig. 5). The boundary of transition of the second Lyapunov factor to the domain of negative values in the system of mutually coupled Rossler oscillators in parameter plane (ω_1, σ) is shown in Fig. 6 (curve *2*). It can be seen that this boundary does not coincide with the boundary of lag synchronization shown in this figure (curve *1*). Moreover, it goes lower than the boundary of lag synchronization and is practically independent of the frequency mismatch between the interacting systems. The fact that the transition of one of the positive Lyapunov exponents in the mutually coupled systems into the domain of negative values is not related with the onset of lag synchronization and that, by analogy with the case of unidirectional coupled systems, it corresponds to the boundary of onset of the regime of generalized synchronization in mutually

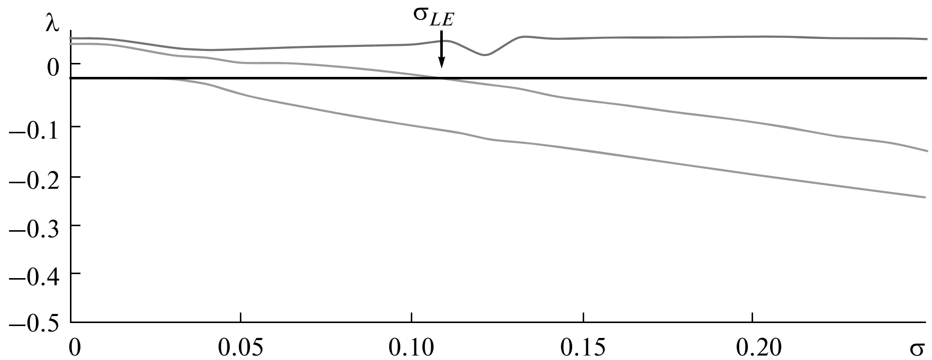

Fig. 5. Four higher Lyapunov exponents vs. coupling parameter σ for the system of coupled Rossler oscillators, $\omega_1 = 0.99$. The instant of transition of one of the positive Lyapunov exponents in the region of negative values σ_{LE} is shown by an arrow.

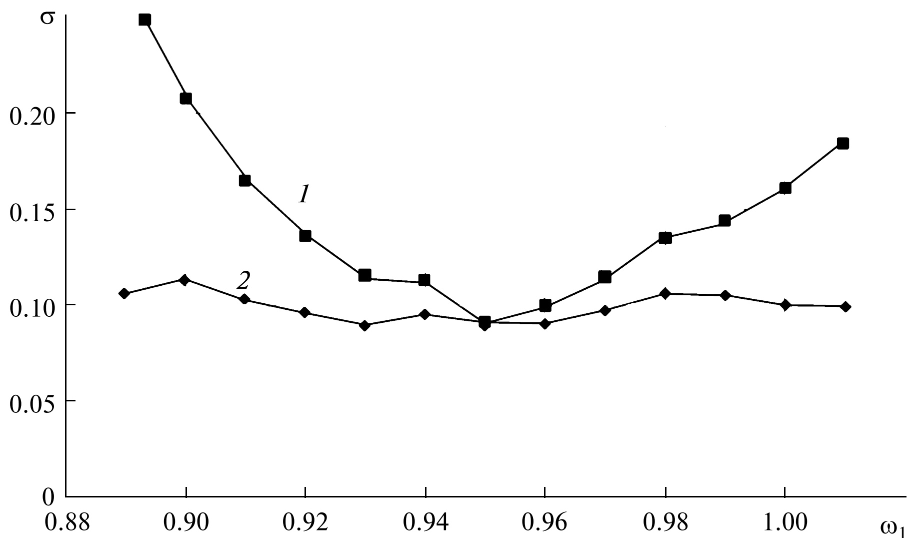

Fig. 6. (Curve *1*) Boundary of the regime of lag synchronization and (curve *2*) the instant of transition of one of the positive Lyapunov exponents to the region of negative values in the system of two mutually coupled Rossler oscillators.

coupled systems was proved in [46] using the nearest neighbor method.

Figure 7a shows the boundaries of generalized synchronization in parameter plane (ω_1, σ) for various values of parameter $\alpha \in [0;1]$. It can be seen from this figure that the boundaries of onset of the synchronous regime in the case of unidirectional and mutual coupling between the systems (curves *1* and *6*, respectively) strongly differ from each other. In the region of relatively large values of the mismatch of eigenfrequencies, the thresholds of generalized synchronization in both cases are close to each other. In the region of relatively small frequency mismatch ($\omega_1 \le 0.91$, $\omega_1 \ge 0.99$), these values rather strongly differ, and, while for identical systems, relationship (13) is approximately satisfied due to the closeness of the regime of generalized synchronization and complete synchronization, the gradual transition from one relationship to the other is observed with increasing mismatch between the systems. This behavior of the boundary of generalized synchronization in unidirectional coupled systems in parameter plane (ω_1, σ) is explained in detail in [45, 47]. Substantial differences in the quantitative values of the threshold of onset of synchronous regime in the region of relatively large and relatively small eigenfrequency mismatches are determined by the differences in the behavior of the main spectral components of the Fourier spectra of interacting systems (see [47, 48] for details). In systems with mutual coupling, the spectral components behave qualitatively in a similar way under variation of parameter ω_1, which does not result in substantial differences in quantitative values of the threshold of onset of generalized synchronization in this case.

One of the most important problems related with the investigation of generalized synchronization is the

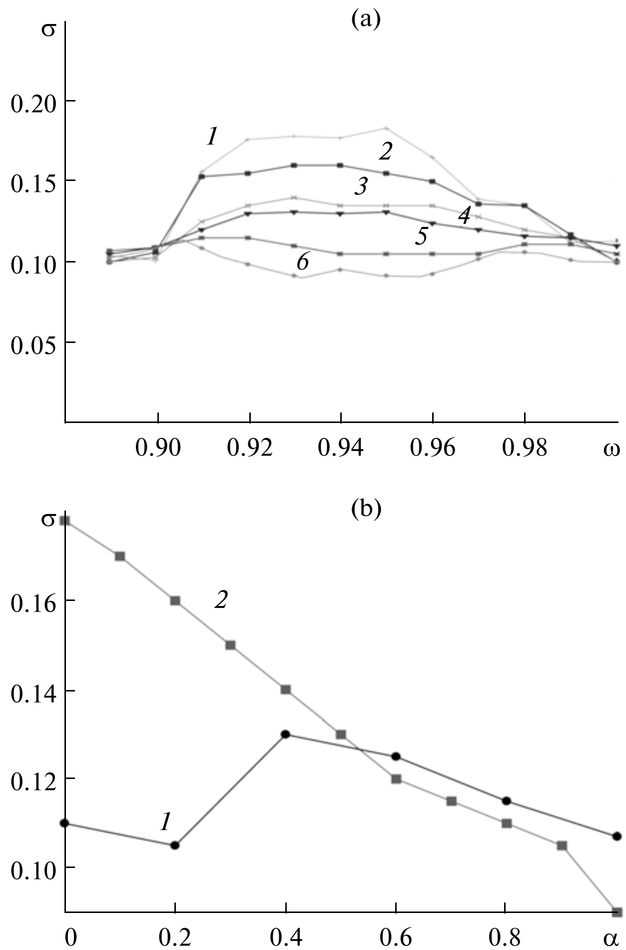

Fig. 7. (a) Boundaries of generalized synchronization of two coupled Rossler systems (14) in parameter plane (ω_1, σ) with increasing α from (curve *1*) 0 to (curve *2*) 1 with a step of 0.2; (b) threshold of generalized synchronization as a function of α for relatively large ($\omega_1 = 0.99$, curve *1*) and relatively small ($\omega_1 = 0.93$, curve *2*) eigenfrequency mismatch.

analysis of the transition from curve *1* to curve *6* (see Fig. 7) and determination of the value of α at which the influence of the frequency mismatch between the systems becomes significant. It can be seen from Fig. 7a that, irrespective of the value of mutual coupling α in the region of relatively large values of eigenfrequency mismatch, the critical values of the coupling parameter are close to each other. In the region of relatively small eigenfrequency mismatch for $\alpha \in [0; 0.6]$, the threshold of onset of the synchronous regime is much higher than the similar value of the coupling parameter in the region of a relatively large frequency mismatch. For $\alpha \in [0.8;1]$, a weak dependence of the threshold of generalized synchronization on the frequency mismatch is observed.

Figure 7b, similar to Fig. 3b, shows the dependence of the threshold of generalized synchronization on parameter α for relatively large and relatively small frequency mismatches. It can be seen that, in the region of small frequency mismatch, the threshold of generalized synchronization almost linearly decreases with increasing α, while, in the region of large frequency mismatch, $\sigma(\alpha)$ exhibits a weak nonlinear dependence.

D. PHASE SYNCHRONIZATION

In this section, we analyze the influence of the degree of the coupling symmetry on the onset of the regime of phase synchronization in system of coupled Rossler oscillators (14). For the chosen values of the control parameters, the attractors of these systems in the absence of coupling are phase-coherent. This makes it possible to introduce the phases of these systems, according to relationship (5), as the angles in the polar coordinate system in planes ($x_{1,2}$, $y_{1,2}$).

Figure 8 shows the boundaries of onset of phase synchronization in parameter plane (ω_1, σ) and the dependences of the threshold values of phase synchronization for the cases of relatively large and relatively small eigenfrequency mismatches. It can be seen that, with increasing α, the threshold of phase synchronization decreases, dependence $\sigma(\alpha)$ is close to a linear function (both in the region of relatively large and relatively small eigenfrequency mismatches).

It can be seen from this plot that, with increasing mismatch, the threshold of onset of phase synchronization in systems with unidirectional coupling begins to grow fast, while, for mutually coupled systems, the growth of the boundary of phase synchronization is much slower. This behavior of phase synchronization in systems with unidirectional coupling can be explained as follows. It is known [49, 50] that, for $\omega_d <$ 0.9 and $\omega_d > 0.98$ in Rossler systems (14) with the above values of control parameters, the onset/destruction of phase synchronization develops via the occurrence/loss of phase coherence of a chaotic attractor. In mutually coupled Rossler systems, according to the calculation, in the entire range of variation of parameter ω_1 shown in Fig. 8, this scenario of onset of phase synchronization is not observed. The differences in the scenario, and therefore, mechanisms of onset of synchronous regimes in systems with unidirectional and mutually coupled systems result in the change of the character of the dependence of the boundary of onset of the synchronous regime.

Thus, for the regime of phase synchronization relationship (13) is approximately satisfied in the bounded range of frequency mismatch. In the region of relatively large eigenfrequency mismatch, relationship (13) is violated for all considered types of synchronous behavior.

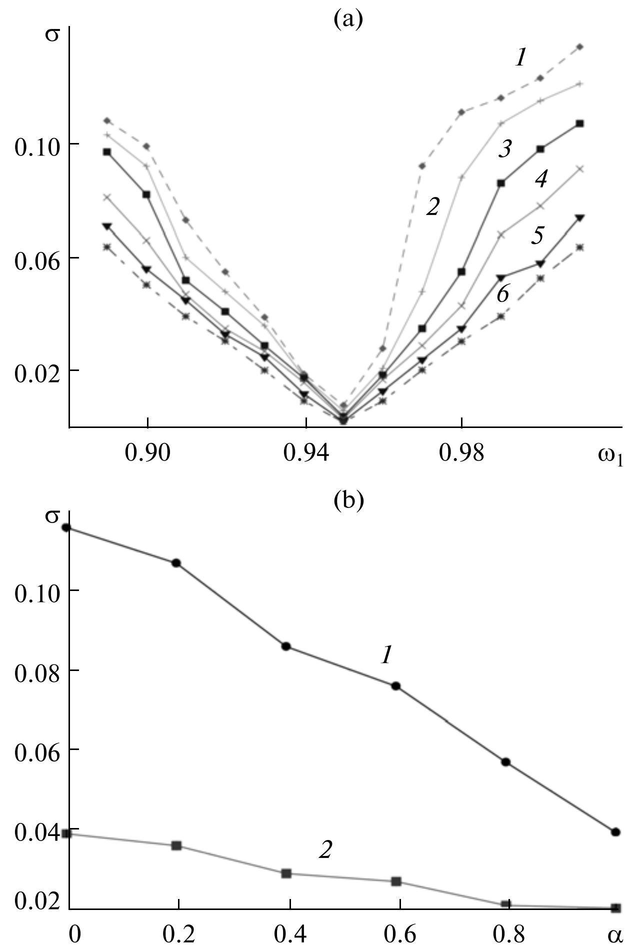

Fig. 8. (a) Boundaries of phase synchronization of two coupled Rossler systems (14) in parameter plane (ω_1, σ) with increasing α from (curve *1*) 0 to (curve *2*) 1 with a step of 0.2; (b) threshold of phase synchronization as a function of α for relatively large ($\omega_1 = 0.99$, curve *1*) and relatively small ($\omega_1 = 0.93$, curve *2*) eigenfrequency mismatch.

CONCLUSIONS

In this paper, we have studied the problem of the influence of the degree of the coupling symmetry on the onset of regimes of complete, lag, phase, and generalized synchronizations. The analytical dependence of the threshold of onset of complete synchronization on the coupling symmetry coefficient has been obtained. In particular, it has been demonstrated that, in two unidirectional and mutually coupled systems, the ratio of the thresholds of its onset is 2 : 1. It has been shown that, for other types of synchronous behavior (lag and phase synchronizations), there is no such strict regularity, and this relationship is approximately satisfied only in the region of relatively small eigenfrequency mismatches of interacting systems. At the same time, in the region of large frequency mismatch, the ratio of the values of the coupling parameter corresponding to the onset of the synchronous regime in unidirectional and mutually coupled systems is much higher.

At the same time, irrespective of the value of the mismatch between the systems for the regimes of lag, complete, and phase synchronizations, the threshold of onset of the synchronous regime is reduced with increasing the coupling symmetry parameter. A similar behavior is observed for generalized synchronization in the region of a relatively small eigenfrequency mismatch. In the region of a large frequency mismatch the threshold of onset of generalized synchronization weakly depends on the coupling mutuality coefficient.

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