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THEORETICAL AND MATHEMATICAL PHYSICS

Influence of Noise on the Behavior of Oscillators near the Synchronization Boundary

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Abstract—Nonautonomous behavior of oscillators in the presence of noise is considered. The influence of noise on the dynamics of local zero Lyapunov exponents for nonautonomous dynamic systems that are near the synchronization boundary is considered. It is shown that the action of noise on a nonautonomous dynamic system that is near the synchronization boundary produces domains of synchronous motion in the series realization, which alternate with asynchronous domains. In accordance with this, the distribution of local zero Lyapunov exponents corresponding to laminar phases shift toward negative values. This effect is demonstrated with a discrete-time system (map of a circle onto itself) that is a reference model to describe the synchronization phenomenon and also with a reference system exhibiting chaotic dynamics (Ressler system).

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INTRODUCTION

In this work, we study the nonautonomous behavior of oscillators near the synchronization boundary in the presence of noise. The urgency of this issue results from several reasons. First, it is obvious that noise is inevitable both in measurements and in numerical simulation. Even if noise can be sometimes neglected, its effect on the system's behavior becomes appreciable near bifurcation points. As a result, the behavior of the system near bifurcation point in the presence of noise and when noise is neglected may differ considerably. It is known, for example, that noise considerably affects the statistic characteristics of intermittent behavior [1–3]. Therefore, it is important to investigate the noise-related behavior of systems and reveal how noise changes quantitative characteristics (e.g., Lyapunov exponents) of the system.

Second, this investigation is intimately related to another topical issue, chaotic phase synchronization [4]. In a number of cases, the chaotic dynamics of a system can be considered as "noisy" periodic oscillations (see, e.g., [5, 6]); accordingly, generality may be expected between the behavior of periodic oscillators near the synchronization boundary in the presence of noise and the behavior of chaotic systems near the phase synchronization boundary. Certainly, the effect of noise on the synchronization boundary has been studied before: it is well known, for example, that noise shifts the synchronous mode threshold (for both periodic and chaotic oscillations) toward higher values of the coupling parameter or external action amplitude [5]. In this case, noise has a destructive effect on the synchronous dynamics of coupled oscillators (or on the synchronous behavior of a nonautonomous system experiencing an external action). At the same time, noise may constructively influence the behavior of a system; specifically, the phenomena of stochastic resonance [7] and noise-induced synchronization are known, the latter being observed both in systems with a small number of degrees of freedom [8] and in spatially distributed systems [9]. Our investigations show [10] that the action of noise on a nonautonomous oscillator that is near the synchronization boundary (when synchronization is not observed yet) may also be constructive and improve the synchronism of oscillations. This is embodied, specifically, in the behavior of one of the zero Lyapunov exponents: in the presence of noise, the zero conventional Lyapunov exponent becomes negative before it goes through the bifurcation value of a control parameter (external signal amplitude or amount of coupling) that sets the synchronous mode under noiseless conditions.

The primary aim of this work is to study the behavior of the zero conventional Lyapunov exponent that governs the behavior of the system near the synchronization boundary in the presence of noise. The structure of this work is the following. In Section 1, the variation of conventional Lyapunov exponents (in the first place, the zero conventional Lyapunov exponent) in the present of noise is described and a model system (map of a circle onto itself) used in further investigations is introduced. In Section 2, we consider local conventional Lyapunov exponents that are responsible for synchronous and asynchronous domains using mapping of a circle in the presence of noise. In Section 3, the exponents are considered using the example of a Ressler system exhibiting chaotic dynamics. Comments and conclusions are drawn in Section 4.

1. ZERO CONVENTIONAL LYAPUNOV EXPONENT

Consider the behavior of two unidirectionally coupled oscillators,

$$\begin{aligned} \dot{\mathbf{x}}_d &= \mathbf{H}(\mathbf{x}_d, \mathbf{g}_d), \\ \dot{\mathbf{x}}_r &= \mathbf{G}(\mathbf{x}_r, \mathbf{g}_r) + \varepsilon\mathbf{P}(\mathbf{x}_d, \mathbf{x}_r), \end{aligned} \qquad (1)$$

where $\mathbf{x}_{d,r}$ are the vectors of state of driving and driven systems, respectively; $\mathbf{H}$ and $\mathbf{G}$ specify the vector field of the systems considered; $\mathbf{g}_d$ and $\mathbf{g}_r$ are vectors of parameters; $\mathbf{P}$ stands for unidirectional coupling between the systems; and parameter ε determines the amount of coupling.

Denoting the dimensions of the phase spaces for the driving and driven systems as N_d and N_r, one can describe the behavior of unidirectionally coupled oscillators (1) with a spectrum of Lyapunov exponents $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{N_d+N_r}$. Since the behavior of the driving system is independent of the state of the driven one, the spectrum of Lyapunov exponents can be subdivided into two parts: the Lyapunov exponents for the driving system, $\lambda_1^d \ge \ldots \ge \lambda_{N_d}^d$, and those characterizing the behavior of the driven system, $\lambda_1^r \ge \ldots \ge \lambda_{N_r}^r$. When coupling parameter ε varies, the Lyapunov exponents of the driving system remain unchanged, since its dynamics in no way depends on the amount of coupling, while the conventional Lyapunov exponents for the driven system vary. Clearly, such an approach can also be applied to describe the nonautonomous behavior of an oscillator experiencing an external action. In this case, it is reasonable to consider only the spectrum of Lyapunov exponents $\lambda_1^r \ge \ldots \ge \lambda_{N_r}^r$, which describe the behavior of the given oscillator, and coupling parameter ε can be viewed as a control parameter determining the external action amplitude. Nevertheless, we will assume that the systems under study are unidirectionally coupled ones, bearing in mind that, if necessary, it is easy to switch to consideration of the nonautonomous behavior of a system that is under an external action.

Importantly, the interacting systems being considered may exhibit both chaotic and periodic dynamics. In the case of chaotic dynamic systems, the leading Lyapunov exponents of each of them (at least λ_1^d and λ_1^r) are positive. In any case, however, if the systems are uncoupled ($\varepsilon = 0$), either Lyapunov spectrum inevitably contains a zero Lyapunov exponent ($\lambda_i^d = 0$ or $\lambda_j^r = 0$, respectively) governing the evolution of a small perturbation that describes the shift of the image point along the phase trajectory in the phase space of a given system. For periodic systems, zero Lyapunov exponents are leading ones (that is, $i = 1$ and $j = 1$).

As coupling parameter ε grows, the zero Lyapunov exponent of the driving system remains zero and the conventional zero Lyapunov exponent characterizing the behavior of the driven system may change. This conventional zero component (designated as Λ_0) is the main subject of investigation in this work. For periodic systems without noise, Λ_0 is known to become negative just at the instant the driven system under study is synchronized by a periodic signal coming to it from the driving system. Therefore, this phenomenon is frequently used as a criterion of synchronous mode establishment and also to find the coupling parameter corresponding to the synchronization boundary.

A somewhat different situation occurs in the presence of noise (no matter whether random or determinate) in the driven system. In this case (see [10]), zero conventional Lyapunov exponent Λ_0 becomes negative before synchronous conditions are set and its value depends on the coupling parameter as

$$\Lambda_0(\varepsilon) \sim \begin{cases} -\dfrac{1}{|\varepsilon - \varepsilon_c|}, & \varepsilon < \varepsilon_c \\ \ln\left|1 - a\sqrt{\varepsilon - \varepsilon_c}\right|, & \varepsilon > \varepsilon_c. \end{cases} \qquad (2)$$

Here, ε is the parameter of coupling between the interacting oscillators, ε_c is the bifurcation value of the coupling parameter (at which synchronous conditions are set in the absence of noise), and a depends on the properties of the systems.

The change of sign of a Lyapunov exponent means, in the general case, that the dynamics of the system changes qualitatively. Sometimes, the fact that one of Lyapunov exponents becomes negative is associated with the occurrence of synchronism, as, for example, in the case of periodic oscillation synchronization or generalized chaotic synchronization [11, 12]. However, when phase synchronization is set in the system of coupled chaotic oscillators, conventional Lyapunov exponent Λ_0 is already essentially negative [6]. Therefore, with regard to the negativity of Lyapunov exponent Λ_0, it can be expected that some attributes of synchronism appear below the phase synchronization boundary although the synchronization conditions are not established yet.

As is known, below the boundary of phase chaotic synchronization, the intermittent behavior is observed [13–15]: the time dependence of phase difference $\Delta\varphi(t)$ for interacting oscillators contains domains of the synchronous behavior (laminar phases of motion, when the phase difference remains nearly the same), which alternate with domains where the absolute value of the phase difference, $|\Delta\varphi(t)|$, changes abruptly by 2π. These domains of the series realization are called turbulent phases. One can therefore expect that the negativity of parameter Λ_0 is due to the presence of synchronous domains (laminar phases).

To check this assumption, let us consider a discrete-time model system—map of a circle onto itself. The mapping dynamics will be analyzed in the interval $x \in [-\pi, \pi)$ with noise ξ_n added to the determinate system,

$$x_{n+1} = x_n + 2\Omega(1 - \cos x_n) - \varepsilon + \xi_n, \mod 2\pi. \quad (3)$$

In (3), ε is a control system that is similar to the coupling parameter in (1), $\Omega = 0.1$ is the frequency detuning of the interacting oscillators, ξ_n is delta-correlated Gaussian noise with intensity D such that $\langle\xi_n\rangle = 0$ and $\langle\xi_n\xi_m\rangle = D\delta(n - m)$.

It is known that the map of a circle is a reference system for describing the phenomenon of synchronization; so, it seems appropriate to use this map in the given case. Synchronization conditions in coupled oscillators (1) are assigned cycles with different periods, and a stable fixed point corresponds to main synchronization conditions 1 : 1. It can be assumed that map (3) is derived from set (1) of coupled periodic oscillators using the Poincaré section method, with variable x_n in map (3) corresponding to the oscillation phases of the driven system at time instants following each other in the oscillation period of the driving system. Since model system (3) is one-dimensional, its behavior is characterized by only one Lyapunov exponent, which plays the role of conventional zero Lyapunov exponent Λ_0 in system (1) of coupled oscillators.

If noise intensity D is zero, saddle–node (tangent) bifurcation takes place in system (3) at $\varepsilon = \varepsilon_c = 0$. In this case, stable, x_s, and unstable, x_u, fixed points merge together at point $x = 0$. Above critical value ε_c ($\varepsilon > \varepsilon_c$), the behavior of system (3) is characterized by fixed stable point x_s (Fig. 1), which corresponds to the synchronous mode in coupled oscillators (1). For ε below the critical value, the behavior of the system features a narrow gap between the map and bisectrix $x_{n+1} = x_n$, so that the image point characterizing the behavior of the system slowly moves along this gap and eventually leaves it. Since the map is considered in the interval $x \in [-\pi, \pi)$ modulo 2π and the segment characterizing the state of the system is closed into a circle, the image point enters into the gap again after a relatively small number of iterations and the process is repeated. Behavior of such a type is called type-I intermittency [16, 17].

In terms of the intermittency theory, the motions of the image point along and outside the gap are viewed as laminar and turbulent phases, respectively. Importantly, in the case of the intermittent behavior ($\varepsilon < \varepsilon_c$) without noise, Lyapunov exponent Λ_0 equals zero, while if control parameter ε exceeds critical value ε_c,

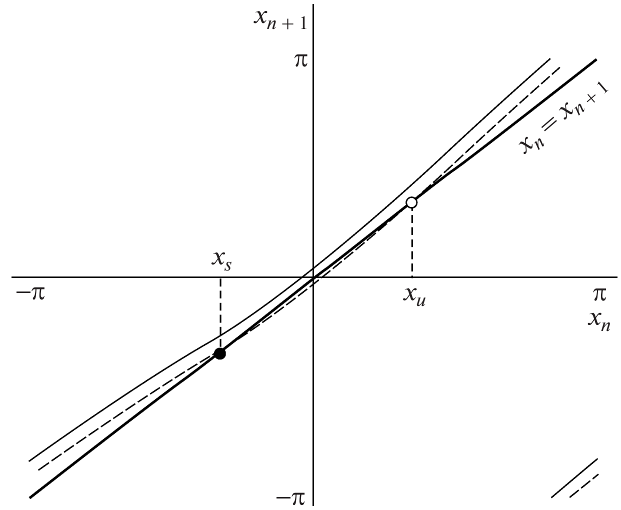

Fig. 1. Map (3) of a circle for $\varepsilon > \varepsilon_c$ (dashed line) and $\varepsilon < \varepsilon_c$ (solid line).

$\Lambda_0 = \ln|1 - 2\sqrt{\Omega\varepsilon}|$. It is easy to see that the Lyapunov component is negative in the latter case, which implies that synchronous conditions are set in flow system (1) of coupled oscillators.

It was shown [10] that, when noise with intensity D is added, Lyapunov exponent Λ_0 becomes negative and, for the map of a circle, is given by

$$\Lambda_0(\varepsilon) \sim \begin{cases} -\dfrac{D\Omega}{4|\varepsilon|}, & \varepsilon < 0 \\ \ln\left|1 - 2\sqrt{\Omega\varepsilon}\right|, & \varepsilon > 0. \end{cases} \quad (4)$$

Note, however, that this expression is inapplicable for extremely low values of parameter ε, i.e., when $\varepsilon \longrightarrow \pm 0$.

Expression (4) becomes valid if [10]

$$D^{2/3} \ll |\varepsilon| \ll 1, \quad \varepsilon < 0 \quad (5)$$

and

$$\varepsilon \gg D^{2/3}, \quad \varepsilon > 0. \quad (6)$$

Figure 2 plots Lyapunov exponents Λ_0 against control parameter ε that were obtained numerically ($D = 4 \times 10^{-6}$ for map (3) and from analytical approximation (4). The same dependence without noise is also shown for comparison. It is seen that the results of numerical simulation are in good agreement with those of analytical approximation (4) throughout the range of ε, except for a narrow domain $\varepsilon \longrightarrow \pm 0$, where these results diverge. In the presence of noise, Lyapunov exponent Λ_0 becomes negative well below the point of saddle–node bifurcation.

As has already been noted, the negativity of Lyapunov exponent Λ_0 may be related to the presence

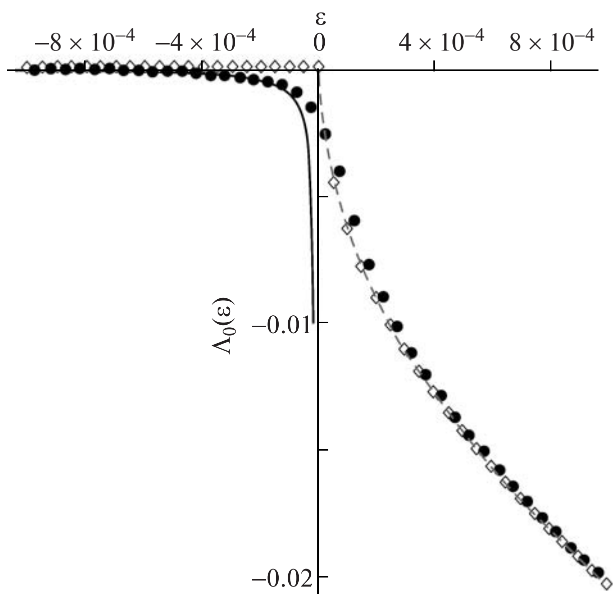

Fig. 2. Lyapunov exponent Λ_0 vs. control parameter ε near the point of saddle–node bifurcation in map (3) of a circle. Data points (◇) and (●)are obtained numerically without noise and in the presence of noise with intensity $D = 4 \times 10^{-6}$, respectively. Solid and dashed lines stand for analytical relationships (4) for supercriticality parameter $\varepsilon < \varepsilon_c$ and $\varepsilon > \varepsilon_c$, respectively.

of laminar phases (synchronization domains for system (1) of coupled oscillators). To substantiate this assumption, we will introduce local Lyapunov exponents (see the next section) and consider them for laminar and turbulent phases separately.

2. LOCAL CONVENTIONAL LYAPUNOV EXPONENTS: MAPPING OF A CIRCLE

To study the dynamics of the system, we will consider local Lyapunov exponents [18, 19] defined for laminar and turbulent phases separately. As a rule, local Lyapunov exponents are determined within a fixed time interval, τ = const. In this case, distribution $N(\lambda_l)$ of local Lyapunov exponents is a very important characteristic of the system's behavior, which is related to Lyapunov exponent λ as

$$\lambda = \frac{1}{N_0}\int_{-\infty}^{\infty}\lambda_l N(\lambda_l)d\lambda_l, \quad (7)$$

where

$$N_0 = \int_{-\infty}^{\infty}N(\lambda_l)d\lambda_l.$$

Since local Lyapunov exponents are used in our investigation in order to characterize the dynamics of the system for either phase separately, each Lyapunov exponent is determined within its own time interval τ. The durations of these intervals differ from each other; therefore, it is necessary to deal with distribution $N(\lambda_l, \tau)$ rather than with $N(\lambda_l)$. Accordingly, relationship (7) should be replaced by

$$\lambda = \frac{\int_{-\infty}^{\infty}d\lambda_l\int_0^{\infty}\lambda_l\tau N(\lambda_l,\tau)d\tau}{\int_{-\infty}^{\infty}d\lambda_l\int_0^{\infty}\tau N(\lambda_l,\tau)d\tau}. \quad (8)$$

Similarly, it is necessary to deal with two distributions $N_s(\lambda_l,\tau)$ and $N_a(\lambda_l,\tau)$ corresponding to synchronous and asynchronous domains in the behavior of the system for laminar and turbulent phases separately. Obviously, Lyapunov exponent Λ_0 is related to these distribution as

$$\Lambda_0 = \frac{\int_{-\infty}^{\infty}d\lambda_l\int_0^{\infty}\lambda_l\tau(N_s(\lambda_l,\tau)+N_a(\lambda_l,\tau))d\tau}{\int_{-\infty}^{\infty}d\lambda_l\int_0^{\infty}\tau(N_s(\lambda_l,\tau)+N_a(\lambda_l,\tau))d\tau}. \quad (9)$$

Local Lyapunov exponents were calculated by the formula

$$\lambda_l = \frac{1}{T_k}\sum_{n=t_k}^{t_k+T_k}\ln|1+2\Omega\sin x_n|, \quad (10)$$

where t_k is the instant of discrete time that corresponds to the beginning of a kth (laminar of turbulent) phase and T_k is the duration of this phase.

Figure 3 shows the distributions of local Lyapunov exponents λ_l for laminar and turbulent phases in the case of map (3) and their corresponding isolines for three different value of noise intensity D. The isolines for distribution $N_a(\lambda_l)$ of local Lyapunov exponents for turbulent phases are solid; those for distribution $N_s(\lambda)$ of local Lyapunov exponents corresponding to laminar phases are shown dashed. In all the distributions, the number of phases is $n = 10^5$. The control parameter of supercriticality was the same, $\varepsilon = -10^{-4}$, in all the cases.

As has been already noted, for the chosen values of the control parameter without noise, map (3) exhibits type-I intermittence and Lyapunov exponent Λ_0 equals zero. In coupled flow oscillators (1) demonstrating the periodic dynamics, such conditions correspond to the behavior that is observed slightly below the control parameter at which the synchronous mode sets in. Clearly, the local Lyapunov exponent distributions for both laminar and turbulent phases are delta-functions in this case ($D = 0$): $N_s(\lambda_l, \tau) = \delta(\lambda_l, \tau - T_s)$ for the local Lyapunov exponent distribution corre-

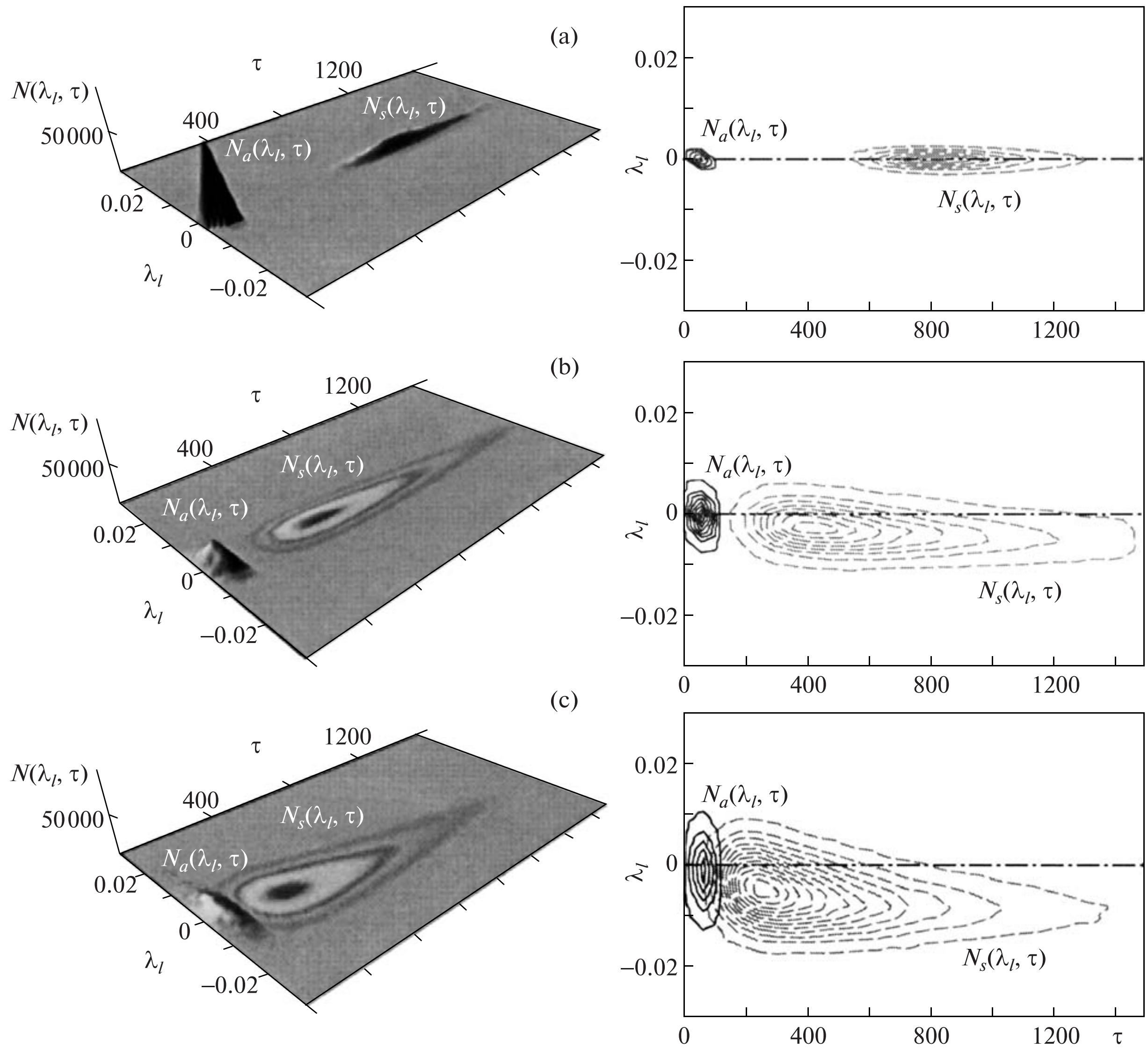

Fig. 3. Distributions of local Lyapunov exponents λ_l for laminar and turbulent phases of map (3) of a circle and their respective isolines for three values of noise intensity D. (a) $D = 10^{-6}$, isolines are shown for turbulent phases with step $h_{N_a} = 2 \times 10^4$ (outer, or minimal, isoline corresponds to $N_a = 2 \times 10^4$) and for laminar phases with step $h_{N_s} = 2 \times 10^3$ (outer, or minimal, isoline corresponds to $N_a = 10^3$), (b) $D = 2.5 \times 10^{-5}$, isolines are shown for turbulent phases with step $h_{N_a} = 5 \times 10^3$ (outer, or minimal, isoline corresponds to $N_a = 10^3$) and for laminar phases with step $h_{N_s} = 4 \times 10^2$ (outer, or minimal, isoline corresponds to $N_a = 10^2$), and (c) $D = 10^{-4}$, isolines are shown for turbulent phases with step $h_{N_a} = 4 \times 10^3$ (outer, or minimal, isoline corresponds to $N_a = 10^3$) and for laminar phases with step $h_{N_s} = 2 \times 10^2$ (outer, or minimal, isoline corresponds to $N_a = 10^2$).

sponding to laminar phases and $N_a(\lambda_l, \tau) = \delta(\lambda_l, \tau - T_a)$ for turbulent phases (T_s and T_a are the durations of the respective phases).

Figure 3a corresponds to the lowest (of those considered in this work) noise intensity, $D = 10^{-6}$. In this case, the dynamics of the system differs from the behavior under the no-noise conditions insignificantly: Lyapunov exponent Λ_0 is very close to zero, $\Lambda_0 = -2 \times 10^{-4}$. At the same time, noise causes distributions $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ to somewhat diffuse, this effect being more pronounced in distribution $N_s(\lambda_l, \tau)$ for laminar phases. The curves diffuse largely along coordinate τ corresponding to the duration of the phases. Importantly, both distributions remain almost symmetric about the τ axis, because of which Λ_0 approaches zero (in accordance with relationship (9)).

As the noise intensity grows (see Fig. 3b, $D = 2.5 \times 10^{-5}$), the local Lyapunov exponent distributions continue to diffuse, this effect being more significant for distribution $N_s(\lambda_l, \tau)$ corresponding to laminar phases. It is seen from Fig. 3b that the heights of the distributions decrease and the distributions become wider. In addition, if local Lyapunov exponent distribution $N_a(\lambda_l, \tau)$ for turbulent phases remains symmetric about the τ axis, which means that the Lyapunov exponent calculated from turbulent phases,

$$\Lambda_{0a} = \frac{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \lambda_l \tau N_a(\lambda_l, \tau) d\tau}{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \tau N_a(\lambda_l, \tau) d\tau}, \quad (11)$$

remains close to zero ($\Lambda_{0a} = -10^{-4}$), the respective distribution for laminar phases, $N_s(\lambda_l, \tau)$, becomes asymmetric and shifts toward negative values. Accordingly, the Lyapunov exponent calculated from laminar phases,

$$\Lambda_{0s} = \frac{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \lambda_l \tau N_s(\lambda_l, \tau) d\tau}{\int_{-\infty}^{\infty} d\lambda_l \int_0^{\infty} \tau N_s(\lambda_l, \tau) d\tau}, \quad (12)$$

will be negative ($\Lambda_{0s} = -3 \times 10^{-3}$), as well as Lyapunov exponent Λ_0, $\Lambda_0 = -2.8 \times 10^{-3}$.

In Fig. 3c, the noise intensity is still higher, $D = 10^{-4}$. The trend observed in Fig. 3b is seen to progress: distributions $N_a(\lambda_l, \tau)$ and $N_s(\lambda_l, \tau)$ continue to widen and lower, the distribution for turbulent phases remains symmetric, and distribution $N_s(\lambda_l, \tau)$ for laminar phases becomes more and more asymmetric and more and more shifts toward negative values. Accordingly, negative Lyapunov exponent Λ_0 increases in absolute value, $\Lambda_0 = -5.2 \times 10^{-3}$. Thus, the negativity of Lyapunov exponent Λ_0 in the map of a "noisy" circle is due primarily to the dynamics of system within laminar phases. It is these phases that are responsible for the negativity of the Lyapunov exponent.

An intriguing consequence following from the above results is that two identical periodic oscillators experiencing a periodic external action that does not synchronize them are expected to demonstrate identical behavior in the presence of noise, while in the absence of noise, their dynamics must be different. This is because the conventional zero Lyapunov exponent Λ_0 is, in the latter case, the leading conventional Lyapunov exponent and its negativity in the presence of noise indicates that noise-induced synchronization is established [8]. To emphasize the nonautonomy of their behavior, it seems appropriate to speak of nonautonomous noise-induced synchronization.

3. LOCAL CONVENTIONAL ZERO LYAPUNOV EXPONENTS: RESSLER SYSTEM

Consider now the behavior of conventional local Lyapunov exponents for a set of unidirectionally coupled chaotic oscillators (1). We take two coupled Ressler systems near the conditions of phase synchronization as such oscillators. In this case, both synchronization and determinate chaotic behavior should be taken into account. The equations of interacting oscillators have the form

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, \\ \dot{y}_d &= \omega_d x_d + a y_d, \\ \dot{z}_d &= p + z_d(x_d - c), \\ \dot{x}_r &= -\omega_r y_r - z_r + \sigma(x_d - x_r), \\ \dot{y}_r &= \omega_r x_r + a y_r, \\ \dot{z}_r &= p + z_r(x_r - c), \end{aligned} \quad (13)$$

where (x_d, y_d, z_d) and (x_r, y_r, z_r) are vectors characterizing the states of the driving and driven oscillators, respectively, and σ is the amount of coupling between the oscillators. The dot denotes time derivatives. The values of other control parameters were the same as in previous works [12, 20]: $a = 0.15$, $p = 0.2$, and $c = 10.0$.

Natural fundamental oscillation frequency ω_r in the driven system was taken to be equal to $\omega_r = 0.95$; the same parameter for the driving system, $\omega_d = 0.93$. For such a choice of control parameters, system (13) of coupled oscillators demonstrates chaotic behavior such that the chaotic attractors of both oscillators are phase-coherent. This allows us to introduce, in a standard manner, phases $\varphi(t)_{d,r}$ of chaotic signals as angles of rotation,

$$\varphi_{d,r} = \arctan\left(\frac{y_{d,r}}{x_{d,r}}\right) \quad (14)$$

on projection planes (x, y) for either oscillator. The presence of chaotic phase synchronization can be found by tracing the time evolution of the phase difference. In the phase synchronization mode, the phase difference must satisfy the phase lock condition [4]

$$|\Delta\varphi(t)| = |\varphi_d(t) - \varphi_r(t)| < \text{const}. \quad (15)$$

It is noteworthy that, although the behavior of oscillators (13) is completely determinate, their chaotic dynamics can be viewed as randomly perturbed periodic oscillations [5, 21]. Once can then expect that the behavioristic characteristics of local Lyapunov exponents revealed in Section 2, which are observed in the presence of noise when the dynamics is regular, will also be observed in the case of determinate systems

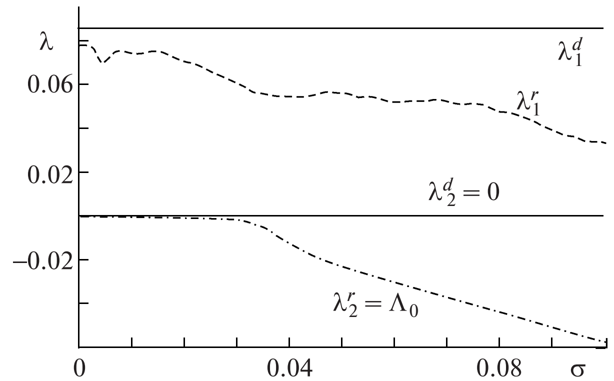

Fig. 4. Lyapunov exponents for set (13) of unidirectionally coupled Ressler oscillators vs. parameter σ. Conventional Lyapunov exponents λ_1^r and λ_2^r are shown by dashed and dash-and-dot lines, respectively.

with chaotic dynamics. The key point here is that the chaotic dynamics cannot be excluded from consideration, unlike noise, the intensity of which can be increased, decreased, or set equal to zero.

One more interesting feature is that the leading Lyapunov exponent for either oscillator from system (13) remains positive unlike the case of mapping a circle (see Section 2). It has been already noted that the behavior of two coupled Ressler systems (13) is characterized by a spectrum of Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_6$, the behavior of the leading system being independent of coupling parameter σ because of the unidirectional coupling between the oscillators. Therefore, the spectrum of Lyapunov exponents can be subdivided into two parts for the driving ($\lambda_1^d > \lambda_2^d > \lambda_3^d$) and driven ($\lambda_1^r > \lambda_2^r > \lambda_3^r$) oscillators separately. The exponents from the first part ($\lambda_1^d > \lambda_2^d > \lambda_3^d$) do not depend on the coupling parameter, while those from the second part ($\lambda_1^r > \lambda_2^r > \lambda_3^r$), which are called conventional exponents, vary with σ.

Figure 4 plots the σ dependences of four Lyapunov exponents (the other two are negatives with a large absolute value, $\lambda_3^d \approx \lambda_3^r \approx -10$, and are of no significance here). Two of the four exponents ($\lambda_1^d > 0$ and $\lambda_2^d = 0$) correspond to the behavior of the driving system; therefore, they are constants. The remaining two exponents, $\lambda_{1,2}^r$, are conventional. In the absence of coupling ($\sigma = 0$), $\lambda_1^r > 0$ (it characterizes the chaotic dynamics of the driven oscillator) and $\lambda_2^r = 0$. Second conventional Lyapunov exponent λ_2^r equals zero and therefore is the main object of this investigation; in

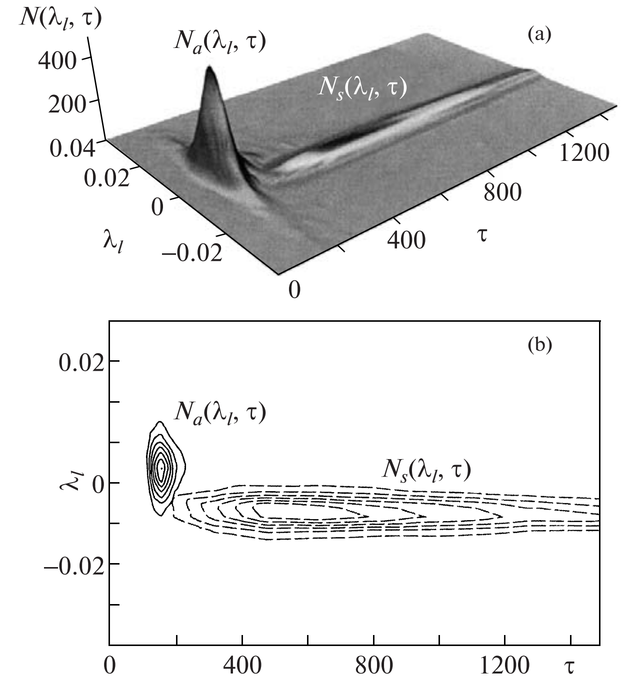

Fig. 5. (a) Distributions of local Lyapunov exponents for the synchronous and asynchronous phases of unidirectionally coupled Ressler oscillators (13) that are constructed from $n = 7125$ phases (the coupling parameter is $\sigma = 0.034$, and conventional Lyapunov exponent $\Lambda_0 = -0.037$. (b) Projections of distributions $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ onto the plane (τ, λ_l). Isolines are shown for turbulent phases with step $h_{N_s} = 70$ (the outer, or minimal, isoline corresponds to $N_a = 70$) and for laminar phases with step $h_{N_a} = 7$ (the outer, or minimal, isoline corresponds to $N_a = 7$).

other words, $\Lambda_0 = \lambda_2^r$. As coupling parameter σ grows, Λ_0 becomes negative and the phase synchronization conditions set in when $\sigma_{PS} \approx 0.039$.

As for the map of a circle (Section 2), the negativity of Lyapunov exponent Λ_0 indicates that some degree of synchronism occurs in the dynamics of the interacting systems, although the phase synchronization mode has not been established yet. As for the map of a circle, let us consider the distributions of local zero exponents $N_s(\lambda_l, \tau)$ and $N_a(\lambda_l, \tau)$ for laminar and turbulent phases separately in the case of two unidirectionally coupled Ressler oscillators (13) (see Fig. 5).

It is seen from Fig. 5 that the local Lyapunov exponents corresponding to the domains of synchronous motion fall into the negative range, while those found for the domains of asynchronous motion (domains where the phase changes abruptly) localize near zero, as in the case of the map of a circle (see Fig. 2).

Thus, it can be concluded that the negativity of the conventional zero Lyapunov exponent is a manifestation of synchronism, which is observed within certain time intervals. It is the domains of synchronous dynamics that are responsible for the negativity of the respective Lyapunov exponent. At the same time, it should be noted that the negativity of the conventional zero Lyapunov exponent in no way means that the conditions of chaotic phase synchronization set in, since there exist domains of asynchronous dynamics, which break laminar phases.

CONCLUSIONS

We considered the influence of noise on the conventional zero Lyapunov exponent for nonautonomous dynamic systems near the synchronization boundary. Local conventional Lyapunov exponents are introduced for laminar and turbulent phases separately. It is shown that the former are responsible for the negativity of conventional zero Lyapunov exponents: the distribution of local conventional Lyapunov exponents on the "conventional Lyapunov exponent–phase duration" plane shifts toward the negative range, while the same distribution for turbulent phases remains nearly symmetric about zero. The negativity of conventional zero Lyapunov exponents is a manifestation of synchronism, which is observed within certain time intervals near the phase synchronization boundary, where the fully synchronous mode has not been established yet.

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