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Theoretical Investigation of the Generalized Synchronization of Dissipative Coupled Chaotic Systems in the Presence of Noise

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Abstract—The noise influence on the generalized synchronization mode in dissipative coupled chaotic systems is analyzed. It is shown that the noise practically does not influence the threshold of the synchronous mode occurrence. The generalized synchronization is noise-resistant. The reasons for the revealed particularity are explained by means of the modified system approach [18] and verified by the results of numerical simulation of unidirectional coupled flow systems and discrete mapping.

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INTRODUCTION

Synchronization of chaotic oscillations represents one of the fundamental phenomena of nonlinear dynamics drawing the steadfast attention of researchers [1, 2]. The interest in this phenomenon is bound up with the fundamental importance of its research and with the wide range of its practical applications, for example, during hidden information channeling; in biological, chemical, and physical problems; and in chaos management, including in systems of microwave electronics [3–5].

Now some types of synchronous behavior of unidirectional and coupled dynamic systems are clearly recognized. Each of the types has fundamental features, for example, phase synchronization, generalized synchronization, time-lagged synchronization, full synchronization, noise-induced synchronization, synchronization of time scales, etc. [1, 6].

One of the most important questions connected with studying the phenomenon of chaotic synchronization is the influence of noise on the transition of synchronous modes. It is known that noise can render both constructive and destructive influence on the behavior of systems. In particular, the influence of external noise can lead to shifting of the threshold valuations of coupling parameters corresponding to the transition of modes of full and phase synchronization [7, 8]. At the same time, general noise can synchronize two not cooperative but identical systems (starting under various initial conditions). In this case, a noise-induced synchronization mode is diagnosed [9, 10].

Now, the noise influence on generalized synchronization [11] is investigated. As an exception, it is possible to note [12] when the question of the noise influence on generalized synchronization in absolutely different unidirectional coupled dynamic systems is studied. It is shown that, in this case, the role of noise is *system-dependent*; i.e., noise can both cause and, on the contrary, destroy a generalized synchronization mode.

For the first time, in the present work, the noise influence on the occurrence of generalized synchronization in dissipative coupled identical chaotic systems with gently detuned parameters is investigated. As shown below, noise practically does not influence the threshold of occurrence of the generalized synchronization mode in such systems; consequently, the generalized synchronization mode is noise-resistant. The identified feature can find application in various areas of science and technology, for example, during hidden information channeling, where the noise level is high enough [13,14].

GENERALIZED SYNCHRONIZATION MODE

The generalized synchronization mode of two unidirectional coupled chaotic oscillators means that, after the transient-process ending between the states of the host $\mathbf{x}(t)$ and slave $\mathbf{u}(t)$ systems, some functional relation $\mathbf{u}(t) = \mathbf{F}[\mathbf{x}(t)]$ is established, which can be complex enough including fractals [11]. To find such relations analytically, as a rule, is not always possible. At the same time, effective diagnostic methods of this mode were discovered, among which the method of auxiliary systems was the greatest [15]. According to this method, along with the slave system, an identical auxiliary system is considered. The initial conditions for the auxiliary system are chosen distinct from the initial state of the slave system but lying in the domain of attraction of the same chaotic attractor. In the case of the absence of a generalized synchronization mode between the cooperative systems, the state vectors of the slave and auxiliary systems belong to the same chaotic attractor but are various due to the Lyapunov instability of the chaotic trajectories. In that case, the generalized synchronization mode takes place by virtue of the functional accuracies fulfilled between the host and slave systems and, respectively, the host and auxiliary systems; after the transient-process' ending, the states of the slave and auxiliary systems should become identical. Thus, the equivalence of the slave and auxiliary systems' states after the transient-process' ending is a criterion of the presence of generalized synchronization between the host and slave chaotic systems.

The analysis of the generalized synchronization mode can also be fulfilled by means of the calculation of the conditional Lyapunov exponents [16]. In this case, the Lyapunov exponents for the slave system are calculated and, as its behavior depends on the state of the host system, they differ from the Lyapunov exponents of the independent system and are called conditional. The existence criterion of generalized synchronization of unidirectional coupled dynamic systems is the highest conditional negative Lyapunov exponent.

The generalized synchronization mode can be observed in systems with various types of coupling: dissipative and not dissipative [16,17]. For dissipative coupled identical dynamic systems with gently detuned parameters, the equations describing the dynamics of the systems can be written down in the following form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{H}(\mathbf{x}(t),\mathbf{g}_d),\\ \dot{\mathbf{u}}(t) &= \mathbf{H}(\mathbf{u}(t),\mathbf{g}_r) + \varepsilon\mathbf{A}(\mathbf{x}(t)-\mathbf{u}(t)),\end{aligned} \quad (1)$$

where $\mathbf{x}(t) = (x_1,x_2,x_3)^T$ and $\mathbf{u}(t) = (u_1,u_2,u_3)^T$ are the state vectors of the host and slave systems, respectively; $\mathbf{H}$ defines the vector field of the examined systems; $\mathbf{g}_d$ and $\mathbf{g}_r$ are the vectors of the parameters; $\mathbf{A} = \{\delta_{ij}\}$ is the coupling matrix; $\delta_{ii} = 0$, $\delta_{ij} = 0$, or $\delta_{ij} = 1$; and ε is the coupling parameter.

For the revelation of the mechanisms leading to the occurrence of the generalized synchronization mode in system (1), we will take advantage of the modified system method [18]. According to this approach, the slave system can be considered as some modified system

$$\dot{\mathbf{u}}_m(t) = \mathbf{H}'(\mathbf{u}_m(t),\mathbf{g}_r,\varepsilon) = \mathbf{H}(\mathbf{u}(t),\mathbf{g}_r) - \varepsilon\mathbf{A}\mathbf{u}(t) \quad (2)$$

with complementary dissipation under the external influence $\varepsilon\mathbf{A}\mathbf{x}(t)$. The generalized synchronization mode arising in system (1) can be examined as a consequence of two interconnected processes proceeding simultaneously: the dissipation increase in the modified system (2) and the increase of the amplitude of the external influence. Both processes are coupled with each other by means of the parameter ε and also cannot be realized in slave system (1) separately from each other. However, the dissipation increase in modified system (2) leads to the simplification of its behavior and the transition from chaotic oscillations to periodic ones. The external influence, on the contrary, complicates the behavior of the modified system and imposes its own dynamics. It is obvious that the occurrence of the generalized synchronization mode is possible only when the chaotic dynamics in the slave system are suppressed by the dissipation.

Thus, the generalized synchronization mode stability is defined, first of all, by the properties of the modified system. The addition of the external noise $\mathbf{B}D\xi(t)$ (where $\mathbf{B}$ is the coupling matrix similar to $\mathbf{A}$) to the equations of system (1) should not lead to an essential change of the modified system's characteristics. If the external noise practically does not change the characteristics of the modified system, the threshold of occurrence of the generalized synchronization mode should not strongly depend on the noise intensity D.

Really, as was mentioned above, the diagnostics of the generalized synchronization mode is possible both by means of the auxiliary system method and by the calculation of the conditional Lyapunov exponents. It is clear that the slave and auxiliary systems can be examined as two identical systems starting from close initial conditions. The calculation of the derivative (in the presence and absence of noise) of their states' difference $\Delta(t) = \upsilon(t) - \mathbf{u}(t)$, in view of the not identity of the determined and stochastic signals influencing these systems, leads to the same equation:

$$\dot{\Delta}(t) = (\mathbf{JH}(\mathbf{u}(t)) - \varepsilon\mathbf{A})\Delta(t) = \mathbf{JH}'(\mathbf{u}(t))\Delta(t), \quad (3)$$

where $\mathbf{J}$ is the Jacobi matrix. As equation (3) can be examined as the equation of the variations at the calculation of the conditional Lyapunov exponents, it is possible to conclude that the highest conditional Lyapunov exponents (defining the threshold of the occurrence of the generalized synchronization mode) will behave similarly in the absence and in the presence of noise. Therefore, the threshold of the occurrence of the generalized synchronization mode should not depend on the intensity of the noise, and the type of synchronous behavior should have noise resistance. Nevertheless, it is necessary to pay attention to that the vector of the slave system state $\mathbf{u}(t)$ in (3), all the same, depends on the random signal $\xi(t)$, and, respectively, noise of high intensity D can change the properties of the modified system, which, finally, can lead to the changing of the boundary of the generalized synchronization mode occurrence.

THE NOISE INFLUENCE ON THE TRANSION OF THE GENERALIZED SYNCHRONIZATION MODE IN SYSTEMS WITH DISCRETE TIME

To verify the correctness of the theoretical arguments stated above, we consider, as the first example, the behavior of unidirectional connected logistic mappings in the case when an external source of noise influences the slave system:

$$\begin{aligned}x_{n+1} &= f(x_n,\lambda_x),\\ y_{n+1} &= f(y_n,\lambda_y) + \varepsilon(f(x_n,\lambda_x) + Df(\xi_n,\lambda_x) - f(y_n,\lambda_y)),\end{aligned} \quad (4)$$

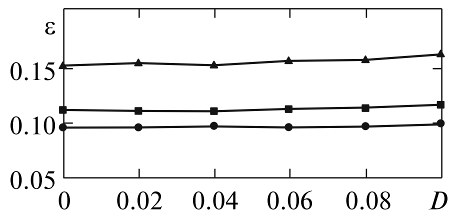

Fig. 1. Dependence of the threshold of the generalized chaotic synchronization mode occurrence in two unidirectional connected logistical mappings (4) on the noise intensity for various values of the control parameters: $\lambda_x = 3.75$, $\lambda_y = 3.75$ (●); $\lambda_x = 3.75$, $\lambda_y = 3.79$ (■); and $\lambda_x = 3.75$, $\lambda_y = 3.90$ (▲).

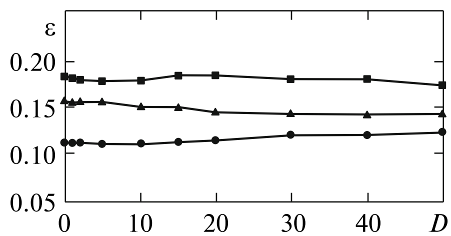

Fig. 2. Dependence of the generalized chaotic synchronization boundary in two unidirectional coupled Roessler oscillators (6) under the influence of the general noise source on its intensity for various values of the parameter ω_x of the host system: $\omega_x = 0.99$ (●), $\omega_x = 0.95$ (■), and $\omega_x = 0.91$ (▲).

where $f(x, \lambda) = \lambda x(1 - x)$ and $\lambda_{x,y}$ are the control parameters of the host and slave systems, respectively; ε characterizes the intensity of the coupling between the oscillators; ξ_n is the random process with regular distributed probability density in the unit interval [0, 1]; and D is the noise intensity. For the diagnostics of the generalized synchronization in system (1), the method of an auxiliary system described in the previous section was used.

The dependence of the threshold of the generalized synchronization mode occurrence on the intensity of the noise D at various values of the control parameters $\lambda_{x,y}$ is given in Fig. 1. From the drawing, it is obvious that the threshold value of the coupling parameter ε practically does not depend on the intensity of the noise in a range of values $D \in [0, 0.1]$. For an explanation of similar behavior of the considered systems, it is necessary to pay attention to the modified system

$$z_{n+1} = (1-\varepsilon)f(z_n, \lambda_y). \tag{5}$$

For the coupling parameter corresponding to the threshold of the generalized synchronization mode occurrence in system (4), the modified system (5) shows periodic oscillations [18]. The external noise's influence practically does not change the characteristics of the slave system and, respectively, practically does not influence the threshold of occurrence of the generalized synchronization mode. During the same time, the subsequent increase of the noise intensity can lead to the changing of the properties of the modified system. In particular, the random influence of a great enough amplitude can reduce the system from a pool of stable cycle attraction, which, in turn, reduces the image point *leaving* to infinity. Nevertheless, the generalized synchronization mode in system (4) is characterized by noise resistance in a wide enough range of noise intensity values D.

NOISE INFLUENCE ON THE TRANSITION OF THE GENERALISED SYNCHRONIZATION MODE IN SYSTEMS WITH CONTINUOUS TIME

As an example, we will consider two unidirectional coupled Roessler systems:

$$\begin{aligned} \dot{x}_1 &= -\omega_x x_2 - x_3 + \varepsilon D\xi, \\ \dot{x}_2 &= \omega_x x_1 + a x_2, \\ \dot{x}_3 &= p + x_3(x_1 - c), \\ \dot{u}_1 &= -\omega_u u_2 - u_3 + \varepsilon(x_1 + D\zeta - u_1), \\ \dot{u}_2 &= \omega_u u_1 + a u_2, \\ \dot{u}_3 &= p + u_3(u_1 - c), \end{aligned} \tag{6}$$

where $\mathbf{x}(t) = (x_1, x_2, x_3)^T$ and $\mathbf{u}(t) = (u_1, u_2, u_3)^T$ are the state vectors of the host and slave systems, respectively; $a = 0.15$, $p = 0.2$, $c = 10$, ω_x, $\omega_u = 0.95$ are the control parameters; the summands $\varepsilon D\xi(t)$ and $\varepsilon D\zeta(t)$ set the external noise influence on the cooperative systems; and $\xi(t)$ and $\zeta(t)$ are the random Gauss process with a zero mean and the dispersion $\sigma = 1.0$. For the integration of system of equations (6), we used the Runge–Kutta method of fourth order adapted for the stochastic differential equations [19] with the time step $\Delta t = 0.001$. The diagnostics of the generalized chaotic synchronization mode were realized by means of the auxiliary system method (see the previous section).

In Fig. 2, the dependence of the threshold of occurrence of the generalized chaotic synchronization mode on the noise intensity for three various values of the control parameter ω_x is given at fixed values of the other control parameters. For the fullest picture, the values of ω_x have been chosen as follows: $\omega_x = 0.99$ is the relatively large frequency detuning of the coupled chaotic oscillators, $\omega_x = 0.95$ are the identical oscillators, and $\omega_x = 0.91$ are the small frequency detunings [20].

It is easy to notice that, independently of the control parameter ω_x, the threshold of occurrence of the generalized chaotic synchronization mode practically does not depend on the intensity of the noise D, and, even for great enough values $D > 20$, the generalized synchronization mode arises approximately at the same values of the coupling parameter ε as in the case without noise. It is shown that, in the considered case, the external noise practically does not change the properties of the modified system's stability:

$$\begin{aligned}\dot{z}_1 &= -\omega_u z_2 - z_3 - \varepsilon z_1,\\ \dot{z}_2 &= \omega_u z_1 + a z_2,\\ \dot{z}_3 &= p + z_3(z_1 - c),\end{aligned} \quad (7)$$

where $\mathbf{z}(t) = (z_1, z_2, z_3)^T$ is the state vector of the modified system. At the specified values of the control parameters in modified system (7), a stable cycle of period one (see also [20]) is realized.

Hence, from the given theoretical consideration, it is obvious that the noise practically does not influence the boundary of the occurrence of the generalized chaotic synchronization mode; consequently, it is possible to speak about the stability of the generalized chaotic synchronization mode under the influence of external noise in systems with a small number of degrees of freedom.

CONCLUSION

For the first time, in the present work, the theoretical and numerical research of the influence of noise on the transition of the generalized synchronization mode is realized for dissipative coupled dynamic systems with gently detuned parameters. It is shown that, independently of the system type and the distribution character of the random variable, in this case, the noise practically does not influence the threshold of the synchronous mode occurrence. The results of the theoretical arguments are verified by the computational modeling of unidirectional coupled flow systems and discrete mapping. At the same time, similar results have been obtained numerically for spatially distributed environments described by the Ginzburg–Landau equations [21] and experimentally for an example of unidirectional coupled radio generators of radio range chaos [22].

The identified features of the boundary behavior of the generalized synchronization in the presence of noise can find practical application in a number of areas of science and technology, for example, at hidden information channeling with a high level of noise [13, 14].

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