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The Threshold of Generalized Synchronization of Chaotic Oscillators

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Abstract—The behavior of two unidirectionally coupled chaotic oscillators exhibited at the threshold of generalized chaotic synchronization is considered. The modified-system approach is applied to explain physical mechanisms of formation of this regime in the cases of large and small mismatches of interacting systems.

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INTRODUCTION

Being a main nonlinear effect important for applications, chaotic synchronization has been studied intensively [1] in recent years. The development of the theory of dynamical chaos has enabled researchers to reveal a great number of various types of chaotic behavior of flow coupled dynamical systems [2–9]: phase synchronization, generalized synchronization, lag synchronization, complete synchronization, and time scale synchronization. Each of these types of synchronous chaotic dynamics exhibits specific features and can be diagnosed by means of specific methods. The possible relationships between these types of synchronous behavior are actively discussed in the literature. Various types of synchronization of chaotic oscillators can be interpreted as various manifestations of common processes developing in coupled nonlinear systems (see, e.g., [8–12]).

Analysis of the relationships between the different kinds of synchronous behavior of coupled chaotic oscillators, in particular, the relationship between generalized synchronization and phase synchronization, is one of the most interesting problems. Initially, phase synchronization was believed to be a weaker type of chaotic behavior [13]. This concept means that, when unidirectionally coupled chaotic oscillators exhibit generalized synchronization, phase synchronization is always observed, while phase synchronization may occur in the absence of generalized synchronization.

However, it was shown later [14] that, depending on the mismatch of the control parameters of coupled chaotic oscillators, phase synchronization may occur at values of the oscillators' coupling parameter that are smaller than those necessary for formation of generalized synchronization.[1] This result means that, on the plane of control parameters, there are regions where generalized synchronization is observed and phase synchronization is not realized. In particular, it has been found for the coupled Rössler systems considered in [14] that, at small mismatches of chaotic oscillators (including the zero mismatch, which corresponds to identical oscillators), the value of the coupling parameter at which generalized synchronization is realized is approximately twice the value corresponding to larger mismatches of control parameters (see also [16]). For the other known types of chaotic synchronization (phase synchronization, lag synchronization, complete synchronization, and time-scale synchronization), the threshold of the synchronous regime (as a function of the mismatch parameter) exhibits antipodal behavior: As the mismatch of the systems' control parameters decreases, the value of the coupling parameter at which the corresponding synchronous regime is formed decreases. Thus, in this context, the regime of complete synchronization differs from the other types of chaotic synchronization. Moreover, this specific feature contradicts the seemingly evident statement that the lesser the mismatch between systems, the easier their synchronization and the smaller the coupling parameter necessary for synchronization.

The purpose of this study is to reveal mechanisms that provide for generalized synchronization of unidirectionally coupled chaotic oscillators.

1. GENERALIZED SYNCHRONIZATION AND THE MODIFIED-SYSTEM APPROACH

Consider interacting unidirectionally coupled drive and response chaotic oscillators $\vec{x}_d(t)$ and $\vec{x}_r(t)$ described by the equalities

$$\begin{aligned} \dot{\vec{x}}_d(t) &= \vec{G}(\vec{x}_d(t)), \\ \dot{\vec{x}}_r(t) &= \vec{H}(\vec{x}_r(t)) + \varepsilon \mathbf{A}(\vec{x}_d(t) - \vec{x}_r(t)), \end{aligned} \qquad (1)$$

[1] Mechanisms responsible for such behavior of coupled oscillators are described in [9, 15].

where $\vec{G}$ and $\vec{H}$ are the operators of evolution of the interacting oscillators, $\mathbf{A} = \{\delta_{ij}\}$ is the coupling matrix, ε is the coupling parameter, $\delta_{ii} = 0$ or 1, and $\delta_{ij} = 0$. In the regime of generalized synchronization [3], the states of these oscillators are related through certain functional dependence $\vec{F}[\cdot]$ such that the functional relationship $\vec{x}_r(t) = \vec{F}[\vec{x}_d(t)]$ becomes valid after the end of the transient process. Dependence $\vec{F}[\cdot]$ may have rather a complicated form necessitating a nontrivial procedure for its determination. Depending on whether the form of functional dependence $\vec{F}[\cdot]$ is smooth or fractal, a strong or weak type of generalized synchronization is formed, respectively [17]. Two different dynamical systems, including systems with different dimensions of the phase space, may serve as interacting oscillators. In addition, generalized synchronization has been described for distributed systems of various nature [18, 19].

Several techniques for diagnosing the regime of generalized synchronization between chaotic oscillators have been proposed in the literature. The nearest neighbor method [3, 20] and the widespread auxiliary system approach [21] are among these techniques. The auxiliary system approach can be described as follows. Response system $\vec{x}_r(t)$ is considered along with auxiliary system $\vec{x}_a(t)$ that is identical to the response system. The initial conditions chosen for auxiliary system $\vec{x}_a(t_0)$ differ from the initial state of response system $\vec{x}_r(t_0)$ but belong to the attraction basin of the same attractor. In the absence of generalized synchronization between interacting systems, the vectors of the response and auxiliary systems, $\vec{x}_r(t)$ and $\vec{x}_a(t)$, belong to the same chaotic attractor but differ. In the presence of generalized synchronization, the relationships $\vec{x}_r(t) = \vec{F}[\vec{x}_d(t)]$ and, accordingly, $\vec{x}_a(t) = \vec{F}[\vec{x}_d(t)]$ hold. Therefore, after the transient process ends, the response and auxiliary systems should become identical: $\vec{x}_r(t) \equiv \vec{x}_a(t)$ (see [21] for details). Thus, the equivalence of the states characterizing the response and auxiliary systems after the end of the transient process (which may be rather long [22]) is a criterion for generalized synchronization between the drive and response oscillators.

In addition, generalized synchronization can be analyzed with the use of calculated conditional Lyapunov exponents [5, 23]. In this case, the Lyapunov exponents are determined for the response system. Since the behavior of the response system is governed by the drive system, these exponents are different from the Lyapunov exponents for the autonomous response system and are referred to as conditional Lyapunov exponents. A criterion for the existence of generalized synchronization in unidirectionally coupled dynamical systems [5, 17] is the negativity of the highest Lyapunov exponent. Furthermore, note that, for unidirectionally coupled chaotic oscillators, the regimes of complete and lag synchronization are particular cases of the generalized-synchronization regime [17].

The modified-system approach [24] can be applied to reveal the causes of generalized synchronization. Consider the behavior of two unidirectionally coupled chaotic oscillators (1). In this case, response system $\vec{x}_r(t)$ can be regarded as a certain modified system

$$\dot{\vec{x}}_m(t) = \vec{H}'(\vec{x}_m(t), \varepsilon) \tag{2}$$

affected by the external signal $\varepsilon\mathbf{A}\vec{x}_d(t)$:

$$\dot{\vec{x}}_m(t) = \vec{H}'(\vec{x}_m(t), \varepsilon) + \varepsilon\mathbf{A}\vec{x}_d(t). \tag{3}$$

Here, $(\vec{H}'(\vec{x}(t)) = \vec{H}(\vec{x}(t)) - \varepsilon\mathbf{A}\vec{x}(t))$.

In this system, the term $\varepsilon\mathbf{A}\vec{x}(t)$ actually introduces additional dissipation to modified system (2). Indeed, the dissipation level and the rate of compression of the phase volume in the system considered are determined by the divergence of the vector field $\operatorname{div}\dot{\vec{x}}$. It is clear that, for the autonomous and modified systems, the relationship

$$\operatorname{div}\vec{H}' = \operatorname{div}\vec{H} - \varepsilon\sum_{i=1}^{N}\delta_{ii} \quad (\varepsilon > 0)$$

holds. Here, N is the dimension of the phase space of the modified system.

Evidently, the generalized synchronization formed in system (1) during an increase in coupling parameter ε can be interpreted as a consequence of two coupled processes that develop simultaneously: an increase in dissipation in modified system (2) and the growth of the external-signal amplitude. Both of the processes are coupled via parameter ε and cannot be realized in response system (1) independently of one another. Nevertheless, it is evident that an increase in dissipation in modified system (2) simplifies its dynamics and results in transition from chaos to periodic oscillations (or, in the case of high dissipation, to the stationary state) [24]. In contrast to this process, in the presence of the external chaotic signal $\varepsilon\mathbf{A}\vec{x}(t)$, the modified system exhibits a more intricate behavior and acquires the dynamics of the external signal. As has been shown in studies [16, 24], generalized synchronization can be formed only when the intrinsic chaotic dynamics of the modified system is suppressed.

Note that generalized synchronization necessitates the existence of a stable periodic regime and the periodic regime is stable owing to the properties of the modified system. In this situation, the equations

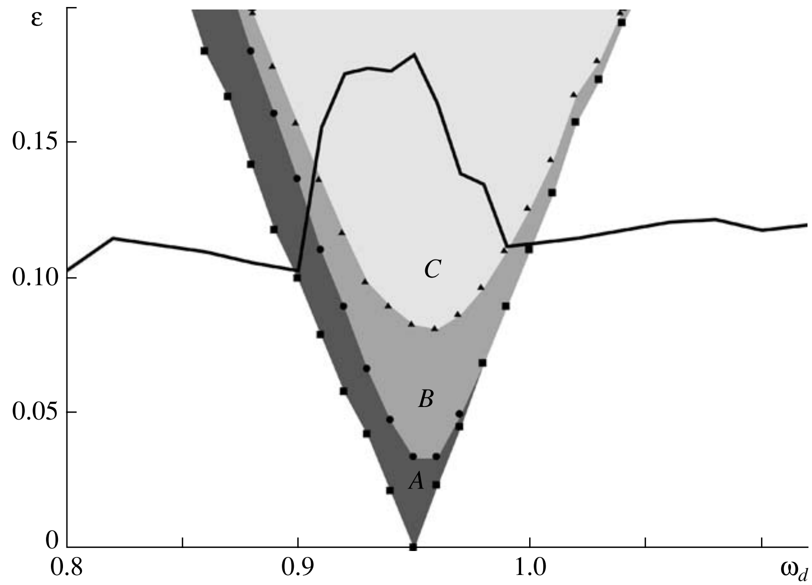

Fig. 1. Threshold of generalized synchronization for two unidirectionally coupled Rössler oscillators (4) and the synchronization tongue of the nonautonomous modified system on the plane (ω_d, ε) of control parameters. Regions *A*, *B*, and *C* correspond to formation of periodic oscillations, a series of period-doubling bifurcations, and chaotization, respectively.

describing modified system (2) do not contain the mismatch of the oscillators' parameters. Therefore, during the analysis of mechanisms that determine the generalized synchronization, it is convenient to fix the values of control parameters $\mathbf{g}_r$ of the response system and change parameters $\mathbf{g}_d$ of the drive oscillator in order to deal with the same values of the control parameters of the modified system.

2. THE THRESHOLD OF GENERALIZED SYNCHRONIZATION IN UNIDIRECTIONALLY COUPLED RÖSSLER SYSTEMS

Consider the onset of generalized synchronization in a system of two unidirectionally coupled Rössler chaotic oscillators whose parameters differ only slightly:

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, \quad \dot{x}_r = -\omega_r y_r - z_r + \varepsilon(x_d - x_r), \\ \dot{y}_d &= \omega_d x_d + a y_d, \quad \dot{y}_r = \omega_r x_r + a y_r, \\ \dot{z}_d &= p + z_d(x_d - c), \quad \dot{z}_r = p + z_r(x_r - c), \end{aligned} \tag{4}$$

where $a = 0.15$, $p = 0.2$, and $c = 10.0$ are control parameters [14] and ε is a coupling parameter. The control parameter of the response system, $\omega_r = 0.95$, which characterizes the main oscillation frequency, is fixed; analogous parameter ω_d of the drive system is varied from 0.8 to 1.1 in order to set the mismatch of the interacting oscillators.

Our investigations have shown that the threshold of generalized synchronization is substantially higher at a small mismatch of the systems than at a large mismatch. At the same time, when the mismatch of the coupled systems is sufficiently large, the values of coupling parameter ε_{GS} at which generalized synchronization is formed is virtually independent of parameter ω_d of the drive system (see Fig. 1).

This behavior of the systems considered can be explained with the use of the modified-system approach presented above (see also [16, 24, 25]). The response system from (4) can be reduced to the modified system

$$\begin{aligned} \dot{x}_m &= -\omega_r y_m - z_m - \varepsilon^* x_m, \quad \dot{y}_m = \omega_r x_m + a y_m, \\ \dot{z}_m &= p + z_m(x_m - c), \end{aligned} \tag{5}$$

where ε^* is the dissipation parameter equal to coupling parameter ε. In order to study the onset of generalized synchronization, we analyze the nonautonomous dynamics of modified system (5),

$$\begin{aligned} \dot{x}_m &= -\omega_r y_m - z_m - \varepsilon^* x_m + F(t), \\ \dot{y}_m &= \omega_r x_m + a y_m, \quad \dot{z}_m = p + z_m(x_m - c), \end{aligned} \tag{6}$$

at the value $\varepsilon^* = 0.11$, which approximately corresponds to the threshold of generalized synchronization in the case of the large mismatch of interacting oscillators. Instead of temporal realization $x_d(t)$ of the drive system from (4), the harmonic signal $F(t) = A\cos(\Omega t)$ with the amplitude $A = \varepsilon B (B \approx 0.105)$ and the frequency $\Omega \approx \omega_d$, which correspond to the dynamics of the drive Rössler oscillator, serves as external signal $F(t)$. This choice is justified by the fact that, at the corresponding values of the control parameter, the Fourier spectrum of drive system (4) has a pronounced main spectral component at the main oscillation frequency.

It has been shown that the behavior of the modified system (and, hence, the behavior of the response sys-

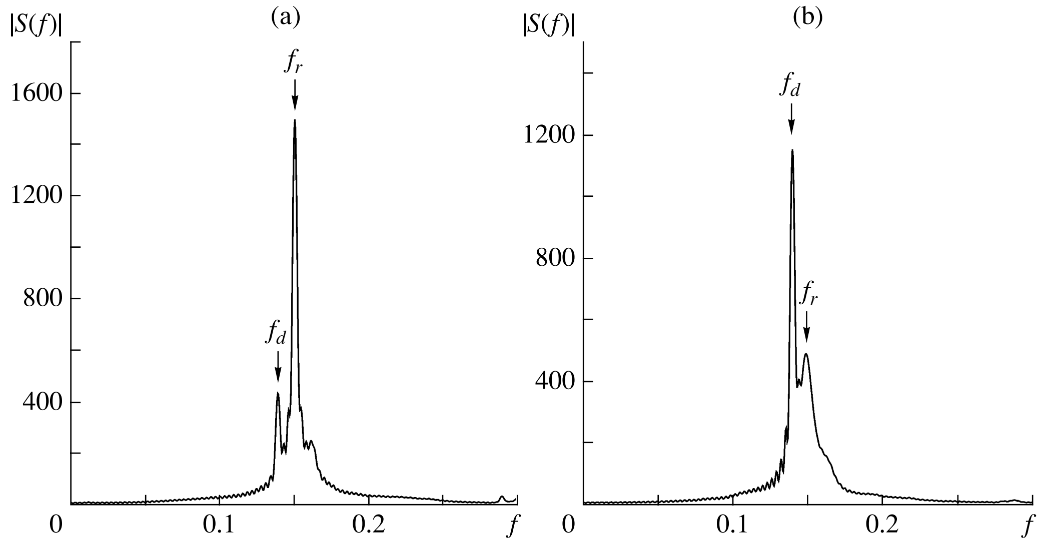

Fig. 2. Fourier spectra of response system (4) that are constructed from a time realization of variable x_r for the coupling parameters ε = (a) 0.09 and (b) 0.14.

tem) should be qualitatively different within and beyond the synchronization tongue. At large mismatches, the threshold of generalized synchronization is determined by the properties of modified system (5) and, hence, independent of parameter ω_d of the drive system. In this situation, the systems are beyond the synchronization tongue and the main frequencies are not locked. At small mismatches, the disposition of the boundary of generalized synchronization is determined by the dynamics of the system within the synchronization tongue. As the amplitude of the external signal within the synchronization tongue of the nonautonomous modified system grows, the system is chaotized via a series of period-doubling bifurcations; i.e., the system exhibits chaotic oscillations that impede generalized synchronization. These oscillations can be suppressed only through an increase in dissipation in the modified system, which corresponds to an increase in coupling parameter ε in the original system of unidirectionally coupled Rössler oscillators. Therefore, the threshold of generalized synchronization shifts to larger values of the coupling parameter. This situation is illustrated in Fig. 1, which displays on plane ($\omega_d = \Omega$, ε) the synchronization tongue of modified Rössler system (5) and the threshold of generalized synchronization in original system (4). Inside the synchronization tongue, the period-doubling lines and the line corresponding to chaos are shown as well.

Thus, the behavior of the threshold of generalized synchronization is determined by the fact that the main natural frequency of the response system is locked by the main frequency of the response system or by the absence of such locking. Within the synchronization region, the boundary of generalized synchronization may shift to larger values of the coupling parameter in the case when the external periodic signal excites the intrinsic chaotic dynamics of the modified system within the synchronization tongue. Hence, the mechanisms determining the generalized synchronization differ within and beyond the region of locking of the main frequencies. Let us consider these mechanisms in more detail.

3. MECHANISM OF GENERALIZED SYNCHRONIZATION REALIZED AT LARGE MISMATCHES OF INTERACTING OSCILLATORS

First, let us consider the case of a large mismatch of the control parameters of interacting chaotic oscillators. When the difference between the main-oscillation natural frequency f_d of the drive system and the main-oscillation natural frequency f_r of the response system is sufficient, these frequencies are not locked and the Fourier spectrum of the response system contains two peaks that correspond to frequencies f_d and f_r. The intensity of these peaks is determined by the coupling parameter. The greater the value of ε, the more pronounced the peak at f_d in the spectrum of the response system and the lower the intensity of the spectral component at f_r (Figs. 2a, 2b). The cause of this effect can be revealed from the analysis of nonautonomous modified system (6): An increase in ε results in the growth of dissipation in the modified system (and, accordingly, in a decrease in the intensities of the spectral components in the system) and, simultaneously, an increase in the amplitude of the external signal, thus leading to an increase in the intensity of the corresponding spectral component.

In study [26], synchronization of the spectral components of chaotic oscillators is described. Thus, two spectral components at frequency f in Fourier spectra

$S_{1,2}(f)$ of interacting systems are synchronized if, after the end of the transient process, the phase difference between these components does not depend on initial conditions. Consider the case when the main spectral components of the Fourier spectra of the partial drive and response systems are not locked (owing to the large mismatch of the control parameters). Let us show that, in this case, generalized synchronization is possible only when, for unidirectionally coupled drive and response systems (1), two incommensurable frequency components ($f_{1d} = f_{1r} = f_1$ and $f_{2d} = f_{2r} = f_2$) of Fourier spectra $S_{r,d}(f)$ of the response ($\vec{x}_r(t)$) and drive ($\vec{x}_d(t)$) oscillators are synchronized. As has been noted above, synchronization of spectral components $f_{1d} = f_1$ and $f_{1r} = f_1$ means that, after the end of the transient process, steady-state phase difference $\Delta\varphi_{f1} \in [0; 2\pi)$ is independent of the initial conditions for the drive and response oscillators [26]. In other words, if Fourier spectra $S_d(f)$ and $S_r(f)$ are calculated from the temporal series of the coupled response and drive systems that start from arbitrary initial conditions, the phase difference between the synchronized frequency components is the same.

Consider two arbitrary phase trajectories $\vec{x}_{1,2r}(t)$ of the response dynamical system that belong to the same chaotic attractor and lie in the time interval $(-\infty; +\infty)$. In this case, we can regard these trajectories as one phase trajectory $\vec{x}(t)$ shifted in time: $\vec{x}_{1r}(t) = \vec{x}(t)$ and $\vec{x}_{2r}(t) = \vec{x}(t+\theta)$, where time shift θ may be arbitrary and, in particular, arbitrarily large. Evidently, Fourier spectra $S_{1,2r}(f)$ corresponding to phase trajectories $\vec{x}_{1,2r}(t)$ are related as follows:

$$S_{1r}(f) = S_{2r}(f)\exp(i2\pi f\theta). \qquad (7)$$

Since the response and auxiliary systems are identical (but start from different initial conditions), the structures of their chaotic attractors are identical and the phase trajectories of the response and auxiliary systems can be represented as $\vec{x}_r(t) = \vec{x}(t)$ and $\vec{x}_a(t) = \vec{x}(t+\theta)$, where time shift θ is generally arbitrary. Hence, in addition, Fourier spectra $S_{r,a}(f)$ corresponding to the phase trajectories of response and auxiliary systems $\vec{x}_r(t)$ and $\vec{x}_a(t)$ satisfy relationship (7), where $S_{1r}(f) = S_r(f)$ and $S_{2r}(f) = S_a(f)$. According to the auxiliary-system method, we can conclude that the condition $\theta = 0$ can be regarded as a criterion for the existence of generalized synchronization.

Now, let us assume that only one pair of spectral components, $f_{d1} = f_{r1} = f_1$, of the Fourier spectra of the drive and response systems turns out to be synchronized. This means that phase difference $\Delta\varphi_{f1}$ between spectral components f_1 of the Fourier spectra of the drive ($S_d(f)$) and response ($S_r(f)$) systems, as well as the phase difference between spectral components f_1 of the Fourier spectra of the drive ($S_d(f)$) and auxiliary ($S_a(f)$) systems, is always the same, while the phase shifts between the remaining spectral components are arbitrary. It can be shown that the set of values of phase shifts θ_n ($n \in \mathbb{Z}$) satisfying the above requirement and corresponding to the phase trajectories of the response ($\vec{x}_r(t)$) and auxiliary ($\vec{x}_a(t)$) systems is countable. Indeed, suppose that the phase difference between frequency components f_1 of the drive and response systems is $\Delta\phi_{f1}$. Then, if time shift θ_n between the phase trajectories of the response and auxiliary system equals the value

$$\theta_n = n/f_1, \quad n \in \mathbb{Z}, \qquad (8)$$

the phase difference between frequency components f_1 of the spectra of the drive and auxiliary systems also equals $\Delta\phi_{f1}$, because $\arg S_r(f_1) = \arg S_a(f_1)$ according to relationship (7). In contrast, if time shift θ does not satisfy relationship (8), the phase difference between frequency component f_{d1} of the drive system and frequency component f_{r1} of the response system differs from the phase difference between frequency component f_{d1} of the drive oscillator and frequency component f_{a1} of the auxiliary oscillator. Thus, if only one pair of components of the Fourier spectra of the drive and auxiliary systems turns out to be synchronized, time shift θ between the phase trajectories of the response and auxiliary systems must satisfy condition (8) and the number of such possible time shifts is infinite. Obviously, in this case, the state vectors of the response and auxiliary systems do not coincide. Hence, generalized synchronization is not observed.

What happens in the case when two pairs of components of the Fourier spectra of the drive and response systems are synchronized? It is evident that, if two multiple (or commensurable) frequencies f_2 and $f_1 = p/qf_2$ ($p, q = 1, 2, 3, \ldots$) in Fourier spectra $S_{r,d}(f)$ of the drive and response oscillators are synchronized, there are an infinite number of possible time shifts between the states of the response and auxiliary systems: $\theta = \theta_2 = n/(f_1)(n = 0, \pm p, \pm 2p, \ldots)$. In the case when two incommensurable spectral components f_1 and f_2 (i.e., the ratio of frequencies f_1/f_2 is an irrational value) are synchronized, there is a unique time shift of $\theta_0 = 0$ between the state vectors of the response and auxiliary systems. Evidently, in this case, the relationship $\vec{x}_r(t) = \vec{x}_a(t)$ holds, thereby indicating the existence of generalized synchronization.

Let us apply the above analysis to study the process of the onset of generalized synchronization in unidirectionally coupled Rössler systems (4). Following the considerations presented in the foregoing, for relatively large mismatches of control parameters in the spectrum of the response system, we separate two spectral components f_d and f_r (Fig. 2). Apparently, the synchronization of these frequency components provides for gener-

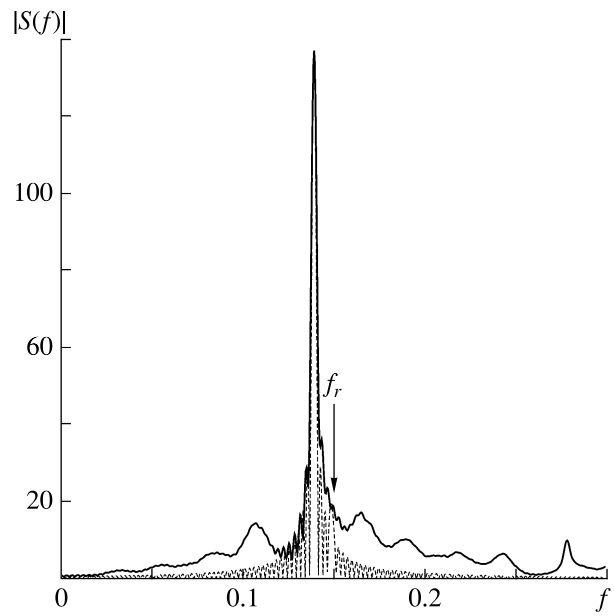

Fig. 3. Fourier spectra of drive system (4) that are obtained from (solid line) a time realization of variable $x_d(t)$ and (dashed line) constructed two-frequency signal $u(t)$. Here, $A_1 = 7.75$, $A_2 = 1.1$, and the arrow indicates frequency f_r. It is seen that amplitudes $A_{1,\,2}$ of signal $u(t)$ are selected such that the intensities of spectral components f_d and f_r are equal in both spectra.

alized synchronization. To verify this supposition, we construct the model signal $u(t) = A_1\cos\omega_d t + A_2\cos\omega_r t$ ($\omega_d = 2\pi f_d$, $\omega_r = 2\pi f_r$), which simulates signal $\vec{x}_d(t)$ of the drive system. Let us show that it suffices to have only these two spectral components in external signal $u(t)$ in order to ensure the coincidence of the states of the response ($\vec{x}_r(t)$) and auxiliary ($\vec{x}_a(t)$) systems ($\vec{x}_r(t) = \vec{x}_a(t)$) affected by this external signal:

$$\begin{gathered}\dot{x}_{r,a} = -\omega_r y_{r,a} - z_{r,a} + \varepsilon(u(t) - x_{r,a}),\\ \dot{y}_{r,a} = \omega_r x_{r,a} + a y_{r,a}, \quad \dot{z}_{r,a} = p + z_{r,a}(x_{r,a} - c).\end{gathered} \tag{9}$$

The aforementioned coincidence is a criterion for the existence of generalized synchronization. We choose amplitudes A_1 and A_2 such that the intensities of harmonic components f_d and f_r (Fig. 3) in Fourier spectrum $S_u(f)$ of signal $u(t)$ match the intensities of the corresponding spectral components in Fourier spectrum $S_{xd}(f)$ of temporal realization $x_d(t)$ of the drive Rössler system. The quantity $A_1 = 7.75$ is determined by the intensity of the main spectral component in spectrum $S_{xd}(f_d)$ of the drive system. The amplitude $A_2 = 1.1$ is determined by the intensity of the chaotic pedestal in spectrum $S_{xd}(f)$ of the drive system at frequency ω_r. The pedestal always exists owing to the chaotic dynamics of the system considered.

Figure 4a shows the phase portrait of nonautonomous response Rössler system (9) affected by a biharmonic signal and (b) plane (x_r; x_a) illustrating the behavior of nonautonomous response and auxiliary systems (9) for the coupling parameter $\varepsilon = 0.11$. It is distinctly seen that all representation points on plane (x_r; x_a) fall on the diagonal $x_r = x_a$, thus indicating the identical behavior of the response and auxiliary systems. In this case, we observe formation of generalized synchronization.

However, the following important aspect of the above analysis should be mentioned. In fact, signal (4) of the drive chaotic oscillator has been replaced with quasi-periodic signal $u(t)$ that has two incommensura-

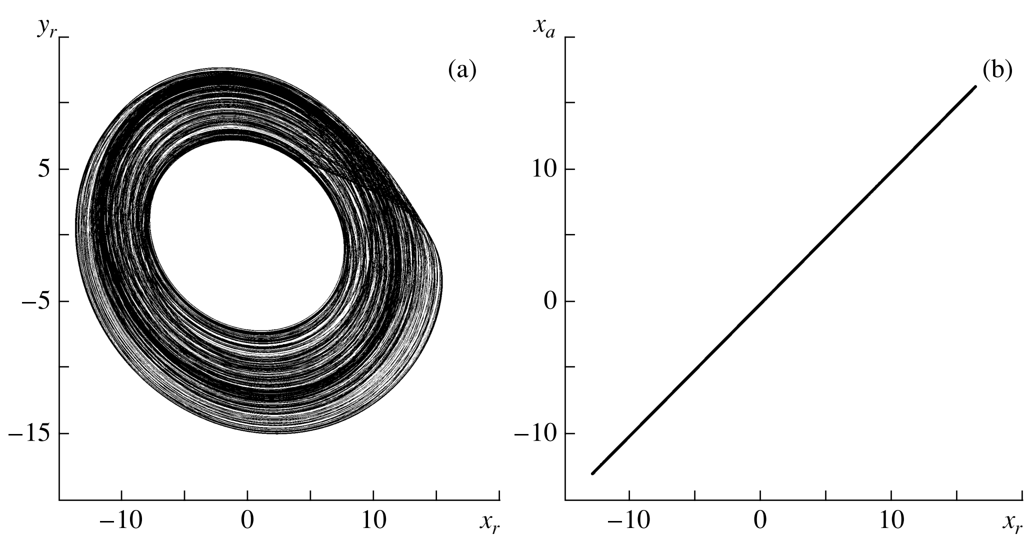

Fig. 4. (a) Phase portrait of nonautonomous response Rössler system (9) and (b) plane (x_r; x_a) illustrating the behavior of nonautonomous response and auxiliary systems (9) for the coupling parameter $\varepsilon = 0.11$.

ble frequencies f_d and f_r. It is clear that the values of the control parameters of the drive oscillator can be chosen such that frequencies f_d and f_r satisfy the rational ratio $f_d^{p,q} = pf_r/q$, where $p, q \in \mathbb{N}$, but the measure of the set of such values of $f_d^{p,q}$ on the real axis equals zero. In the absence of an external signal (e.g., if $A_1 = A_2 = 0$), system (9) exhibits a periodic behavior at the chosen value of parameter ε (see Section 2). This result means that, at the chosen values of the control parameters, an external quasi-periodic signal affects the system exhibiting periodic oscillations. Therefore, response Rössler system (9) is synchronized by external quasi-periodic signal $u(t)$, which simulates a temporal realization of the drive Rössler system, in the generalized-synchronization regime at values of coupling parameter ε that are lower than those in the case of generalized synchronization of system (4). Actually, in the case considered, the threshold of generalized synchronization of the Rössler system by a quasi-periodic signal is close to the threshold of steady-state period-1 periodic oscillations in modified system (5), i.e., at $\varepsilon \approx 0.07$.

It is clear that, in the case of generalized synchronization of Rössler systems (4), the signal affecting the response system is chaotic. Signal $u(t)$ constructed above adequately simulates the energy composition of the main components of the Fourier spectrum of temporal realization $x_d(t)$ that provide for the steady-state synchronous dynamics but does not reproduce all the features of the drive system's signal. At the same time, apparently, the presence of a chaotic pedestal in spectrum $S_{xd}(f)$ of the drive system determines the growth of the value of coupling parameter ε at which generalized synchronization occurs. In order to verify this supposition, let us add signal $mv(t)$ which models the chaotic component of signal $x_d(t)$ to quasi-periodic signal $u(t)$. Parameter m is the amplitude of added signal $v(t)$, and signal $v(t)$ is constructed from temporal realization $x_d(t)$ through removal of spectral components f_d and f_r from the spectrum of $x_d(t)$. In our simulations, signal $v(t)$ is formed according to the following procedure. Fourier transform $\Phi[\cdot]$ is applied to temporal realization $x_r(t)$ generated by drive Rössler system (4). In the obtained spectrum $S_{xr}(f) = \Phi[x_r(t)]$, it is assumed that $|S_{xr}(f^*)| = 0$, where $f^* \in [f_d - \Delta f_d; f_d + \Delta f_d] \cup [f_r - \Delta f_r; f_r + \Delta f_r]$, $\Delta f_r = 10^{-3}$, and $\Delta f_d = 2.5 \times 10^{-3}$. Quantities Δf_d and Δf_r are chosen to be different, because of the different intensities of the corresponding spectral components and their different widths, respectively. After that, the inverse Fourier transform $v(t) = \Phi^{-1}[S'_{xr}(t)]$ is applied to obtained spectrum $S'_{xr}(f)$ (Fig. 5a). Thus, using the signal $w(t) = u(t) + mv(t)$ and varying parameter m in the interval [0; 1], we can simulate the effect of chaotic dynamics on the threshold of generalized synchronization in system (4): at $m = 0$, temporal realization $w(t)$ is quasi-periodic and coincides with $u(t)$, while at $m = 1$, it exhibits characteristics similar to those of temporal realization $x_d(t)$. The corresponding situation is illustrated in Figs. 5b–5d, where the spectra of signal $w(t)$ are displayed at various values of parameter m. Figure 5b shows the spectrum of two-frequency signal $u(t)$ corresponding to the case $m = 0$ for signal $w(t)$. The spectra of signal $w(t)$ for the parameter values $m = 0.5$ and 1.0 are depicted in Figs. 5c and 5d. Assume that the parameter value $\varepsilon = 0.09$ is fixed and system (9) is affected by signal $w(t)$ (instead of $u(t)$) with various values of parameter m. Then, we see that generalized synchronization is formed at the value $m < 0.3$.

Thus, we can conclude from the foregoing that, for relatively large mismatches of unidirectionally coupled chaotic oscillators, generalized synchronization is formed primarily owing to synchronization of two spectral components with incommensurable frequencies. One of these frequencies is the main frequency of the drive system. Evidently, this component is always synchronized with the spectral component at the same frequency in the Fourier spectrum of the response system (i.e., with the response to the component at the main frequency of the drive system). The other frequency is the response system's natural frequency synchronized by the spectral component of the chaotic pedestal of the Fourier spectrum of the drive system. However, the remaining frequency components of the chaotic pedestal of the Fourier spectrum of the drive system likewise play an important role. The effect of these components determines the value of coupling parameter ε at which generalized synchronization occurs.

4. MECHANISM OF GENERALIZED SYNCHRONIZATION REALIZED AT SMALL MISMATCHES OF INTERACTING OSCILLATORS

Now, let us consider mechanisms that determine generalized synchronization at small mismatches of the control parameters of unidirectionally coupled chaotic oscillators (1). As has been shown in Section 2, in this case, the disposition of the boundary of generalized synchronization is determined by locking of the main spectral components. It has been also shown that, as coupling parameter ε and, accordingly, the amplitude of the external signal increase, nonautonomous modified system (6) passes to chaos via a series of period-doubling bifurcations (see also Fig. 1). Thus, mechanisms providing for generalized synchronization differ in different regions of the synchronization tongue and, hence, should be investigated independently.

As has been shown above, generalized synchronization occurs in original system (4) at small mismatches when the coupling parameter satisfies the inequality $\varepsilon > 0.15$ (see Fig. 1). At such values of the parameters, the nonautonomous modified system with $\varepsilon^* = 0.11$ exhibits its chaotic behavior. However, separation of the mechanisms resulting in steady-state generalized synchroni-

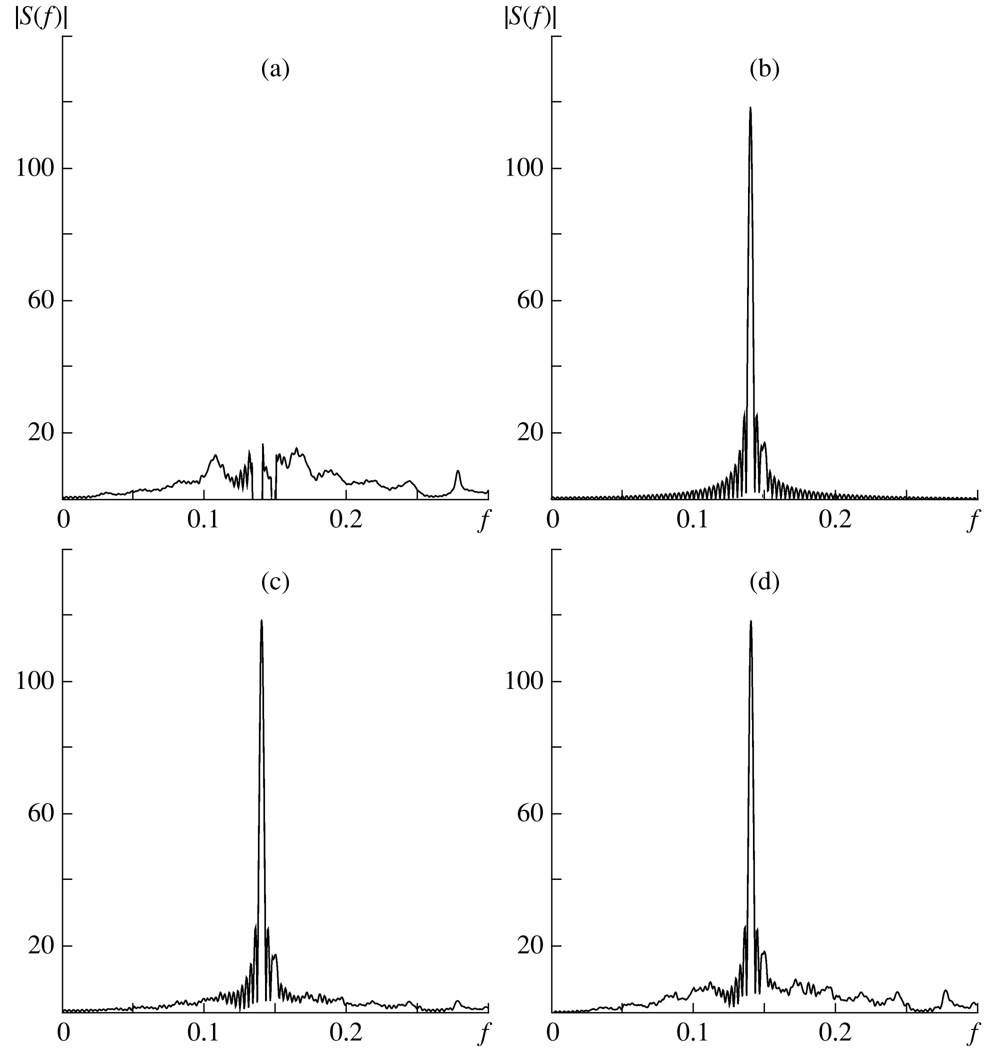

Fig. 5. Fourier spectra of signals (a) $v(t)$ and $w(t)$ for m = (b) 0, (c) 0.5, and (d) 1.0. The differences of the corresponding spectra shown in Figs. 3 and 5d result from the different numbers of averaging procedures performed to construct these spectra.

zation (corresponding to relationship (6) and the equality $F(t) = \varepsilon x_d(t)$) ensures the coincidence of the states of the response ($\vec{x}_r(t)$) and drive ($\vec{x}_a(t)$) systems in the presence of the external signal produced by the drive system even in the case when the corresponding autonomous modified system (5) exhibits periodic oscillations.

Let us start the consideration with the elementary case when the nonautonomous modified system exhibits periodic oscillations with the period-1 cycle. The Fourier spectrum of this system contains one distinct spectral component at the frequency equal to the natural frequency of the system's oscillations. Then, the evolution of the system is independent of the initial conditions. Thus, in order to synchronize two identical systems that start from different initial conditions, it suffices to apply a periodic external signal (providing for periodic oscillations with the period-1 cycle in the response and auxiliary systems). Since the frequency of the external signal is close to the natural frequency of the modified system, frequency locking and, hence, synchronization occur (see also [27]). It is clear that, in the case of chaotic generalized synchronization, the external signal affecting the response system is chaotic. However, this effect does not result in destruction the synchronization mechanisms described above.

When the nonautonomous modified system exhibits periodic oscillations with the period-2 cycle, the set of the initial conditions from which the response system starts under fixed initial conditions of the auxiliary system can be separated into two regions: In the presence of a harmonic external signal, either in-phase or antiphase oscillations are excited in the response and auxiliary systems. Evidently, in the first case, synchronization is always observed (as in the case of periodic oscillations with the period-1 cycle), while the second case calls for additional analysis and is of the most interest. Let us consider this case in more detail.

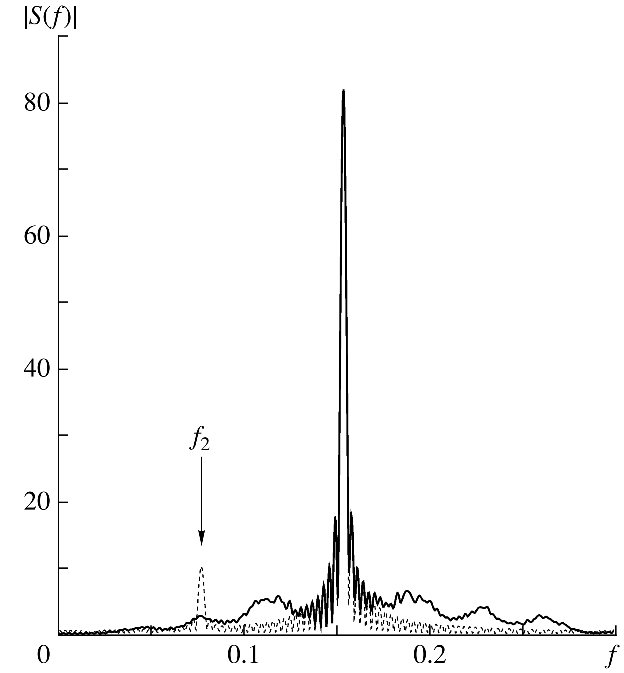

Fig. 6. Fourier spectra of drive Rössler system (4) that are obtained from (solid line) a time realization of variable $x_d(t)$ and (dashed line) artificial signal $u(t)$. Here, $A_1 = 4.5$ and $A_2 = 0.568$. Amplitude A_1 of signal $u(t)$ is selected such that the intensities of spectral components f_d are equal in both spectra. Amplitude A_2 substantially exceeds the value of the noise pedestal in the Fourier spectrum of the drive system. The arrow indicates the corresponding subharmonic $f_2 = \Omega_2/(2\pi)$. At the chosen values of parameters, generalized synchronization is observed in system (9).

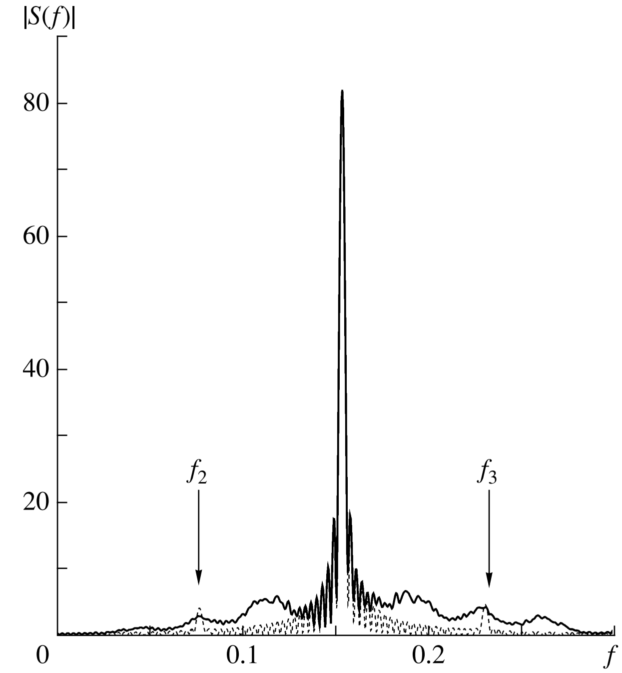

Fig. 7. Fourier spectra of drive Rössler system (4) that are obtained from (solid line) a time realization of variable $x_d(t)$ and (dashed line) artificial signal $u(t)$. Here, $A_1 = 4.5$, $A_2 = 0.236$, and $A_3 = 0.264$. It is seen that amplitudes $A_{1,\,2,\,3}$ of signal $u(t)$ are selected such that the intensities of spectral component f_d are equal in both spectra and the amplitudes of (arrows) the remaining two harmonics, $f_2 = \Omega_2/(2\pi)$ and $f_3 = \Omega_3/(2\pi)$, do not exceed the value of the noise pedestal in the Fourier spectrum of the drive system. At the chosen values of the parameters, generalized synchronization is observed in system (9).

Since the nonautonomous modified system exhibits periodic oscillations with the period-2 cycle, its Fourier spectrum contains subharmonics in addition to the spectral component corresponding to the main frequency of the system's oscillations. It is natural to suppose that generalized synchronization is realized in this case owing to locking of these spectral components. First, we analyze the behavior of system (9) in the case when signal $u(t)$ is a sum of two signals, i.e.,

$$u(t) = A_1\cos(\Omega t) + A_2\cos(\Omega_2 t),$$

where $\Omega = 2\pi f_d$ and A_1 are the frequency and amplitude of the harmonic signal chosen such that the main spectral components of this signal and of the drive system from (4) have identical amplitudes and identical frequencies and $\Omega_2 = \Omega/2$ and A_2 are the frequency and amplitude of the auxiliary signal corresponding to a subharmonic. When quantity A_2 changes in the interval $[0; A_1]$, it is possible to obtain generalized synchronization at amplitude A_2 exceeding the value $A_2 \simeq A_1/8$. The Fourier spectra of this signal and of the drive system are displayed in Fig. 6. It is seen that the amplitude of the second harmonic substantially exceeds the value of the noise pedestal in the Fourier spectrum of the drive system.

Now, consider the case

$$u(t) = A_1\cos(\Omega t) + A_2\cos(\Omega_2 t) + A_3\cos(\Omega_3 t),$$

where $\Omega_3 = 3/2\Omega$; i.e., one more component is added to the external signal. Let us vary A_2 and A_3 until the states of the response ($\vec{x}_r(t)$) and auxiliary ($\vec{x}_a(t)$) systems coincide. Indeed, addition of the third harmonic allows generalized synchronization in the case when the amplitudes of added harmonics do not exceed the value of the noise pedestal in the Fourier spectrum of the drive system (see also Fig. 7). Evidently, completion of model signal $u(t)$ with harmonics corresponding to higher frequencies will improve the considered situation. A similar case is observed for the parameter values at which period-2^nT cycles are realized in autonomous system (5) until chaos resulting in the collapse of generalized synchronization in unidirectionally coupled oscillators (4) occurs within the synchronization tongue of nonautonomous modified system (6).

Thus, we can conclude that the mechanisms determining the steady-state generalized synchronization in a system of two unidirectionally coupled chaotic oscillators with slightly mismatched control parameters are determined by synchronization of the main spectral component of the response system and its subharmonics.

CONCLUSIONS

In this study, we have analyzed the causes of steady-state generalized chaotic synchronization and have explained the character of disposition of the synchronous-regime boundary on the parameter mismatch–coupling coefficient plane of the control parameters. The modified-system approach has been applied to interpret the physical mechanisms of generalized synchronization. The mechanisms of formation of the synchronous regime are different for large and small mismatches of interacting systems. In the case of large mismatches, the existence of generalized synchronization is determined by the properties of the modified system. In this case, synchronization occurs owing to locking of two spectral components with incommensurable frequencies. These components correspond to the main frequency of the drive system and the natural frequency of the response system. The remaining frequency components contained in the chaotic pedestal of the Fourier spectrum of the drive system affect the threshold of generalized synchronization. At relatively small mismatches of interacting systems, the mechanisms of steady-state generalized synchronization are determined completely by the synchronization of the main spectral component of the response system and its subharmonics.

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