

Are generalized synchronization and noise-induced synchronization identical types of synchronous behavior of chaotic oscillators?

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Abstract

This Letter deals with two types of synchronous behavior of chaotic oscillators—generalized synchronization and noise-induced synchronization. It has been shown that both these types of synchronization are caused by similar mechanisms and should be considered as the same type of the chaotic oscillator behavior. The mechanisms resulting in the generalized synchronization are mostly similar to ones taking place in the case of the noise-induced synchronization with biased noise.

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Synchronization of chaotic oscillators has been intensively investigated recently. The chaotic synchronization plays an important role for the analysis of physiological and medicine data, for a chaotic communication, etc. [1–6]. Traditionally, different types of synchronous behavior of chaotic oscillators are distinguished. Each of them is characterized by its own features and may be detected by specific methods which are different for every synchronous regime [1,2]. The important aim of research is finding the regularities of the chaotic synchronization regimes and detecting a relationship between them [7–9]. In particular, we have shown [9,10] that the different types of the chaotic synchronization behavior of the flow systems (such as phase synchronization, lag synchronization, generalized synchronization, complete synchronization) may be considered as one type of the synchronous dynamics, namely, time scale synchronization. Obviously, it is important to develop the further generalization of the chaotic synchronization theory to detect

the common mechanisms resulting in arising the synchronous behavior.

The aim of this work is to show that the two types of synchronous behavior of the chaotic oscillators (the generalized synchronization [11–17] and the noise-induced synchronization [18–31]) which are traditionally supposed to be different are caused by the same mechanisms and should be considered as one phenomenon.

The *generalized synchronization* regime (GS) in two unidirectionally coupled chaotic oscillators means the presence of a functional relation $\mathbf{u}(t) = \mathbf{F}[\mathbf{x}(t)]$ between the state vectors of the drive $\mathbf{x}(t)$ and the response $\mathbf{u}(t)$ systems [11,12]. This relation may be rather complicated and the method of detecting it is usually non-trivial. Depending on the character of this relation $\mathbf{F}[\cdot]$ —smooth or fractal—GS is divided into the strong and the weak ones [12], respectively. It is also important to note that the distinct dynamical systems (including the systems with the different dimension of the phase space) may be used as the drive and response oscillators.

To detect the generalized synchronization regime the auxiliary system approach [14] may be used. In this case the behavior of the response system $\mathbf{u}(t)$ is considered together with the auxiliary system $\mathbf{v}(t)$ one. The auxiliary system is equivalent

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to the response one, but the initial conditions must be different, i.e., $\mathbf{v}(t_0) \neq \mathbf{u}(t_0)$, although both $\mathbf{v}(t_0)$ and $\mathbf{u}(t_0)$ have to belong to the same basin of chaotic attractors (if there is the multistability in the system). If GS takes place in the unidirectionally coupled chaotic oscillators, the system states $\mathbf{u}(t)$ and $\mathbf{v}(t)$ become equivalent after the transient is finished due to the existence of the relations $\mathbf{u}(t) = \mathbf{F}[\mathbf{x}(t)]$ and $\mathbf{v}(t) = \mathbf{F}[\mathbf{x}(t)]$. Thus, the coincidence of the state vectors of the response and the auxiliary systems $\mathbf{u}(t) \equiv \mathbf{v}(t)$ is considered as a criterion of the GS regime presence.

The generalized synchronization regime may also be detected by means of the conditional Lyapunov exponent calculation [12]. GS arises in the system of two unidirectionally coupled chaotic oscillators only if the highest Lyapunov conditional exponent is negative [12].

The *noise-induced synchronization* [18–21,26,27] means that two identical non-coupled chaotic oscillators $\mathbf{v}(t)$ and $\mathbf{u}(t)$ are driven by the common external noise $\xi(t)$. The external noise may result in the consistence of the vector states of the considered systems after the transient is finished. The noise-induced synchronization as well as GS may be realized only if all conditional Lyapunov exponents are negative [32–34].

It has been shown in earlier articles that it is not always possible to observe the noise-induced synchronization in chaotic oscillators, because in this case the chaotic system must display particular properties in the phase space (large contraction region, limited expansion region, and a permanence time that within the expansion region is greater than in the contraction region) [29,30]. At the same time it is necessary to emphasize that biased noise is not a pure noise-induced transition, and therefore contraction regions in that case do not play a crucial role.

It is known that there are two similar mechanisms causing noise-induced synchronization appearance: (i) the external noise signal $\xi(t)$ has the mean non-zero value that results in “moving” the system to the non-chaotic regime [35–40]. In this case the states of the dynamical systems follow the external noise $\xi(t)$ in the same way, and, accordingly, they coincide with each other; (ii) the external noise with the large amplitude (perhaps, with the zero mean value) moves the image point corresponding to the system state to the region of the phase space with the strong dissipation. In other words, the external noise allows the system to spend more time in the region of the phase space where the convergence of the phase trajectories takes place [27–31,41–45]. So, in both cases the convergence of the phase trajectories and, correspondingly, the phase flow contraction, play the main role in the noise-induced synchronization appearance. One can say, that the noise-induced synchronization is caused by introducing the additional dissipation into the system either by means of the bias of the noise or with the help of the large noise amplitude.

The similar effects concerning introducing the additional dissipation in the system result in the generalized synchronization regime appearance. As it has been shown in our works [46,47], there are also two mechanisms causing the GS existence. The first of them is realized if GS takes place in two systems with unidirectional dissipative coupling. For such situation the equations describing the system dynamics may be

written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{H}(\mathbf{x}(t)), \\ \dot{\mathbf{u}}(t) &= \mathbf{H}(\mathbf{u}(t)) + \varepsilon \mathbf{A}(\mathbf{x}(t) - \mathbf{u}(t)),\end{aligned}\quad (1)$$

where $\mathbf{A} = \{\delta_{ij}\}$ is the coupling matrix, ε is the control parameter characterizing the coupling strength between the chaotic oscillators, $\delta_{ii} = 0$ or 1, $\delta_{ij} = 0$ ($i \neq j$). In this case one can see that the response system $\mathbf{u}(t)$ may be considered as a *modified system*

$$\dot{\mathbf{u}}_m(t) = \mathbf{H}'(\mathbf{u}_m(t), \varepsilon) \quad (2)$$

(where $\mathbf{H}'(\mathbf{u}(t)) = \mathbf{H}(\mathbf{u}(t)) - \varepsilon \mathbf{A}\mathbf{u}(t)$) under the external force $\varepsilon \mathbf{A}\mathbf{x}(t)$:

$$\dot{\mathbf{u}}_m(t) = \mathbf{H}'(\mathbf{u}_m(t), \varepsilon) + \varepsilon \mathbf{A}\mathbf{x}(t). \quad (3)$$

It is easy to see that the term $-\varepsilon \mathbf{A}\mathbf{u}(t)$ brings the additional dissipation into the system (2). Indeed, the phase flow contraction is characterized by means of the vector field divergence. Obviously, the vector field divergences of the modified and the response systems are related with each other as

$$\text{div } \mathbf{H}' = \text{div } \mathbf{H} - \varepsilon \sum_{i=1}^N \delta_{ii} \quad (4)$$

(where N is the dimension of the modified system phase space), respectively. So, the dissipation in the modified system is greater than in the response one and it increases with growth of the coupling strength ε .

The generalized synchronization regime arising in (1) may be considered as a result of two cooperative processes taking place simultaneously. The first of them is the growth of the dissipation in the system (2) and the second one is an increase of the amplitude of the external signal. Evidently, both processes are correlated with each other by means of parameter ε and cannot be realized in the coupled oscillator system (1) independently. Nevertheless, it is clear, that an increase of the dissipation in the modified system (2) results in the simplification of its behavior and the transition from the chaotic oscillations to the periodic ones. Moreover, if the additional dissipation is large enough the stationary fixed state may be realized in the modified system. On the contrary, the external chaotic force $\varepsilon \mathbf{A}\mathbf{x}(t)$ tends to complicate the behavior of the modified system and impose its own dynamics on it. Obviously, the generalized synchronization regime may not appear unless own chaotic dynamics of the modified system is suppressed.

One can see that in this case the reasons resulting in the generalized synchronization arising are very similar to the mechanisms which may be revealed for the noise-induced synchronization with biased noise. Indeed, as well as in the case of the biased noise the system state is moved by the deterministic effect to the non-chaotic regime and, as result, the generalized regime may be detected.

The second mechanism of GS arising is realized when two oscillators are coupled in the unidirectional non-dissipative way. In this case the signal of the master oscillator should be introduced with the large amplitude into the response system.

This signal moves the response system state in the region of the phase space with the strong dissipation (see, e.g., [46]) as well as in the case of the noise-induced synchronization. Both mechanisms of GS arising are characterized by the convergence of the phase trajectories and all conditional Lyapunov exponents are negative in these cases. It should be noted that in [47] it was shown that both mechanisms lead to the GS regime onset simultaneously.

So, one can see, that the noise-induced synchronization and the generalized synchronization regimes are caused by the same mechanism. In most cases this mechanism is the suppression of own chaotic dynamics of the response system by means of the non-zero mean of the noise, or with the help of the additional dissipative term, or by moving the system state into the regions of the phase space with the strong convergence of the phase trajectories. It should be noted, that it is not a rigorous mathematical proof, but the given arguments seem to be quite convincing for understanding the unified character of these two phenomena.

The equivalence of these two types of the synchronous behavior may also be illustrated by the following conclusion: the noise-induced synchronization regime means the presence of the functional relationship $\mathbf{F}[\cdot]$ between the chaotic oscillator state and the stochastic signal. Indeed, two identical systems $\mathbf{u}(t)$ and $\mathbf{v}(t)$ driven by the common stochastic force $\xi(t)$ in the regime of the noise-induced synchronization behave equivalently, i.e., $\mathbf{u}(t) = \mathbf{v}(t)$. Obviously, $\mathbf{u}(t) = \mathbf{F}_u[\xi(t)]$ and $\mathbf{v}(t) = \mathbf{F}_v[\xi(t)]$, where $\mathbf{F}_u[\cdot]$ and $\mathbf{F}_v[\cdot]$ are some functional dependences, distinct for the different initial conditions. Nevertheless, in the noise-induced synchronization regime after the transient is finished the vector states of considered systems coincide with each other, therefore, $\mathbf{F}_u[\cdot] \equiv \mathbf{F}_v[\cdot] \equiv \mathbf{F}[\cdot]$ independently on the initial conditions. So, in the case of the noise-induced synchronization the following functional relation takes place: $\mathbf{u}(t) = \mathbf{v}(t) = \mathbf{F}[\xi(t)]$. The same statement is used for the generalized synchronization definition, when the response system is driven by the chaotic signal instead of the stochastic one.

Let us show, that the generalized synchronization regime may be obtained if the drive chaotic system is replaced by the noise signal. This effect may also be treated as the noise-induced synchronization. As the first example of such system behavior let us consider the unidirectionally coupled logistic maps

$$\begin{aligned} x_{n+1} &= f(x_n), \\ y_{n+1} &= f(y_n) + \varepsilon(f(x_n) - f(y_n)), \end{aligned} \quad (5)$$

where $f(x) = ax(1-x)$, a is the control parameter, ε is the coupling strength. The presence of the GS regime in this system for some values of the coupling strength ε has been shown (see [12]). Let us consider now the behavior of the response system y_n when the dynamics of x variable is not determined by the dynamical system (5), but it is the stochastic process ξ_n which is characterized by the probability distribution $p(\xi)$. In this case the dynamics of the response system is described by

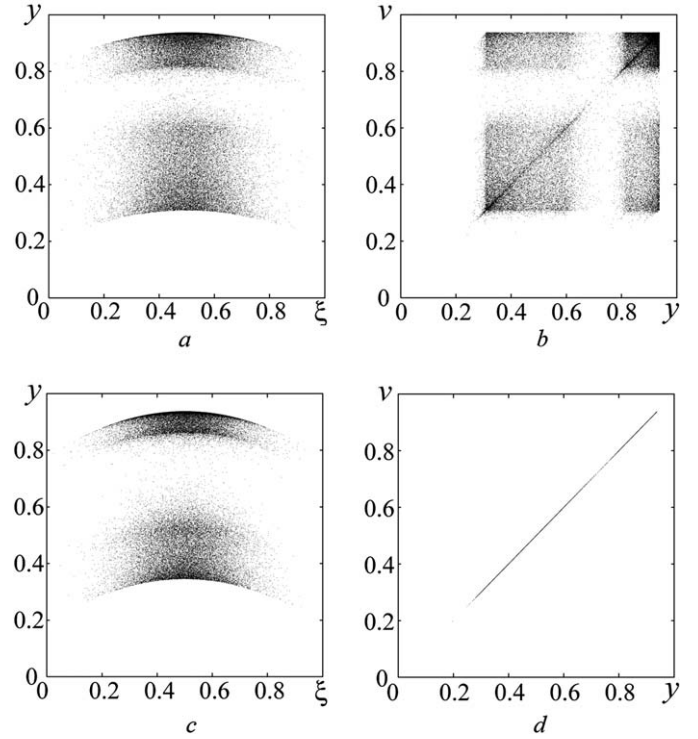


Fig. 1. The planes (ξ_n, y_n) and (y_n, v_n) of the logistic maps for the coupling strength $\varepsilon = 0.125$ (a), (b), and $\varepsilon = 0.175$ (c), (d). It is clear that in the case (d) the response y_n and the auxiliary v_n systems demonstrate the identical behavior $y_n = v_n$ that testifies the presence of the functional relationship $y_n = \mathbf{F}[\xi_n]$, and, therefore, the establishment of the synchronization regime.

the equation

$$y_{n+1} = f(y_n) + \varepsilon(f(\xi_n) - f(y_n)). \quad (6)$$

We have shown that the synchronous dynamics between the stochastic process and the state of the dynamical system can also take place in spite of the random character of ξ as well as in the cases of the generalized synchronization or the synchronization induced by the noise. This effect is very similar to the noise induced synchronization with biased noise although the movement of the system state into non-chaotic regime is caused by the term $-\varepsilon f(y_n)$ instead of the bias of noise.

To detect the presence of the relationship between the stochastic process ξ_n and the state y_n of the dynamical system we have used the auxiliary system approach described above. The behavior of the response and the auxiliary systems is shown in Fig. 1(b) when the parameters have been selected as $a = 3.75$ and $\varepsilon = 0.125$, the probability distribution of the random variable ξ_n is

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\xi - \xi_0)^2}{2\sigma^2}\right), \quad (7)$$

where $\xi_0 = 1/2$, $\sigma = 0.11$.¹

It is clear, the response and the auxiliary systems are characterized by the different states in the same moment of discrete

¹ It is important to note that the character of the distribution of the random variable ξ does not matter and the similar results may be observed for the others types of the probability distribution $p(\xi)$, for example, for the uniform one.

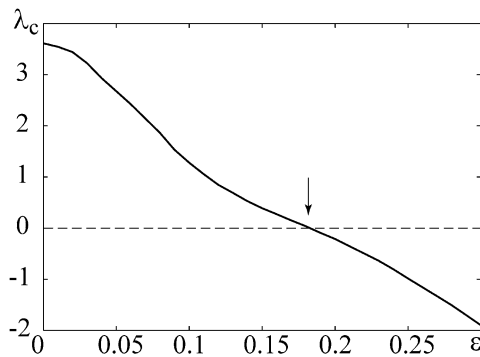
time when the coupling strength is small enough ($\varepsilon = 0.125$). The points corresponding to the states of the response and the auxiliary systems are spread over all area (y_n, v_n). It means that there are no functional relation between the stochastic process ξ_n and the state y_n of the dynamical system.

With increasing the coupling strength ($\varepsilon = 0.175$) the behavior of the considered system is radically changed (see Fig. 1(d)). The points corresponding to the state of the considered systems are on the straight line $v_n = y_n$. Therefore, the relationship $y_n = F[\xi_n]$ takes place and the synchronous behavior is observed. It is important to note, that the functional relationship $F[\cdot]$ is fractal (see Fig. 1(c)) that corresponds to the case of the weak synchronization [12]. Nobody can detect the presence of the functional relationship between ξ_n and y_n taking into account (ξ, y)-plane only (compare Fig. 1(a) when the

synchronous regime is not observed and Fig. 1(c) when the synchronization takes place, respectively).

The presence of the synchronous regime is also confirmed by the dependence of the conditional Lyapunov exponent λ_c on the coupling strength ε (Fig. 2). One can see that λ_c is positive for the small values of the coupling parameter, therefore, there is no the functional relationship between ξ_n and y_n . When the coupling strength increases the conditional Lyapunov exponent λ_c becomes negative, therefore, the synchronous regime is detected and the relationship $y_n = F[\xi_n]$ between stochastic process ξ_n and the state y_n of the logistic map (6) takes place.

The analogous results have been obtained for the Rössler system under the external stochastic signal. As in the case of the first example (6) let us replace the dynamics of the drive system by the stochastic process ξ_n . The investigated system has the following form:



$$\begin{aligned} \dot{x}_r &= -\omega_r y_r - z_r + \varepsilon(\xi_n - x_r), \\ \dot{y}_r &= \omega_r x_r + a y_r, \\ \dot{z}_r &= p + z_r(x_r - c), \end{aligned} \tag{8}$$

where $a = 0.15$, $p = 0.2$, $c = 10.0$, $\omega_r = 0.95$ are the control parameter values, $\xi_0 = 0$, $\sigma = 11.2$ are mean value and dispersion of probability distribution function (7) of random value ξ_n , respectively. As well as for the logistic map with noise term (see Eq. (6)) the observed effect is similar to the noise-induced synchronization with biased noise.

Fig. 2. The dependence of the conditional Lyapunov exponent λ_c of the system (6) on the coupling strength ε . The stochastic signal is characterized by the normal distribution (7), the onset of the synchronization is marked by an arrow.

For $\varepsilon = 0.05$ (see Fig. 3(a), (b)) the noise-induced synchronization does not observed, i.e., all points on (x, v)-plane characterizing response and auxiliary systems states are spread

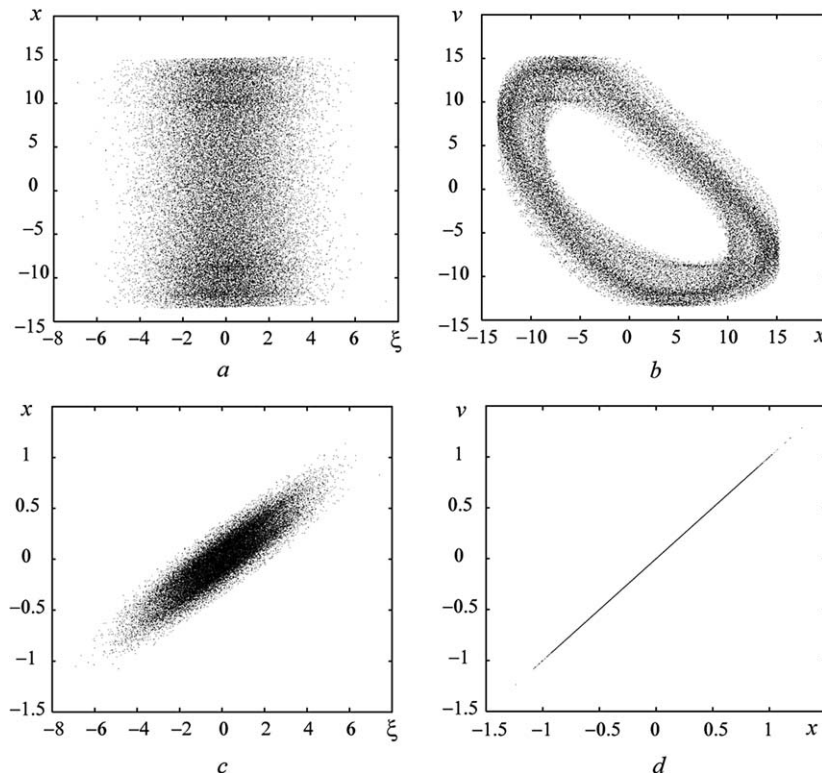


Fig. 3. The planes (ξ_n, x) and (x, v) of the Rössler systems for the coupling strength $\varepsilon = 0.05$ (a), (b), and $\varepsilon = 0.15$ (c), (d). It is clear that in the case (d) the synchronization regime is observed.

randomly. When the coupling parameter increases ($\varepsilon = 0.15$), the response and auxiliary systems demonstrate identical behavior (see Fig. 3(d)). This situation also corresponds to the case of weak GS synchronization (see Fig. 3(c)). It should be noted that the external stochastic signal has been introduced in system (8) in the way that is typical for the mutually coupled oscillators when the GS regime takes place. Alternatively, this coupling term is not typical for the system where noise-induced synchronization is observed. We think that this example is an additional argument confirming our conclusion.

In conclusion, we argue that the generalized synchronization and the noise-induced synchronization regimes are caused by the same mechanism. As it has been mentioned above, this mechanism is the suppression of own chaotic dynamics of the response system by means of introducing the additional dissipation. The additional dissipation may be introduced into the system either by means of the mean non-zero value of the noise, or with the help of the additional dissipative term, or by moving the system state into the regions of the phase space with the strong convergence of the phase trajectories. Typically, the mechanisms resulting in the generalized synchronization act like ones in the case of the noise-induced synchronization with biased noise when the system state is moved (by means of the dissipative term or biased noise) to the non-chaotic regime. Nevertheless, the other mechanism corresponding to the movement of the system state into the regions of the phase space with the strong dissipation by means of the external signal with large amplitude or by means of large zero-mean noise may also take place (see, e.g., [12,27]).

So, the difference between the generalized synchronization and the noise-induced synchronization is only in character of the driving signal. In case of the noise-induced synchronization the stochastic signal drives the chaotic oscillator, while in the case of the generalized synchronization the signal of another chaotic dynamical system is used. That is why the system with the different dimensions of the phase space may be used to obtain the generalized synchronization regime. Obviously, the identity of the system is not required in this case and, in general, the driving signal may be arbitrary. Although the generalized synchronization and the noise-induced synchronization are traditionally distinguished as different types of the synchronous behavior, it may be appropriate and useful to consider them as one type of the synchronous behavior caused by one reason.

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