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STABILITY OF THE SYNCHRONOUS STATE OF AN ARBITRARY NETWORK OF COUPLED ELEMENTS

S. Boccaletti,[1] A. A. Koronovsky,[2] D. I. Trubetskov,[2] A. E. Khramov,[2] and A. E. Khramova[2] *

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We propose a method for determining the range of the coupling parameter for which the network of slightly nonidentical chaotic oscillators demonstrates stable synchronous behavior. As an example of using this method, we study the complete-synchronization regime of a network of nonidentical Rössler oscillators.

Over the past decade, there has been active development of the theory of the so-called complex networks whose structure is irregular and the elements demonstrate chaotic behavior [1, 2]. Interest in a study of such networks is related to both the necessity of analyzing various natural, social, and engineering objects and the importance of revealing fundamental aspects of chaotic synchronization in a system of many coupled partial subsystems [3–6]. Special attention is paid to the networks characterized by high variability in the power of interelement coupling [1].

The necessity of studying such networks is also stipulated by the nonlinear-antenna technique, which is now actively developed [7–9]. The active module of a nonlinear antenna is developed on the basis of a network of coupled elements demonstrating chaotic behavior [7, 8]. The nonlinear-antenna technique is based on use of coupled nonlinear elements such as, e.g., radioengineering oscillators, as an active antenna module [8, 10, 11]. Varying the control parameters of a network of coupled oscillators and the interelement coupling, we can observe either simultaneous existence of different synchronous clusters or complete synchronization in the network. The latter allows us to obtain the specified directional pattern [8]. An important issue is to ensure long-term constancy of the directional-pattern shape in the presence of external and internal noise. In other words, the observed regime of complete synchronization should be stable to small perturbations. Therefore, we should indicate the stability boundaries for a synchronous state of the entire network on the whole. It is worth noting that conventional methods for calculating the stability boundaries for the problem in question are almost useless [12]. For example, the standard procedure of plotting the Lyapunov-exponent maps for a complex network which can comprise several tens of thousands of coupled elements involves calculation of the same number of senior Lyapunov exponents, which requires huge computation time even if modern computers are used.

At present, there exists a method for diagnosing the synchronous-regime stability of a network comprising any number of interacting identical elements, which is based on considering the largest Lyapunov exponent [12, 13]. However, considering actual objects, e.g., the active module of a nonlinear antenna, we should reject such an idealization as identity of the network elements. In this paper, we present a method allowing us to diagnose the synchronous state of stability of a network of coupled nonidentical oscillators with slightly nonidentical parameters. Slight nonidentity is understood as such a difference in the control-parameter values that does not result in changes in the dynamical regime observed for a free element of the network. In other words, we assume that the considered dynamical regime is rough.

* rabbit@nonlin.sgu.ru

[1] CNR–Istituto dei Sistemi Complessi, Florence, Italy; [2] N. G. Chernyshevskii State University of Saratov, Saratov, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 49, No. 10, pp. 917–924, October 2006. Original article submitted July 6, 2006; accepted October 24, 2006.

Let us consider a network consisting of N coupled dynamical systems with slightly nonidentical control parameters:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i, \mathbf{g}_i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}[\mathbf{x}_j], \tag{1}$$

where $i = 1, 2, \ldots, N$, $\mathbf{x}_i$ is the state vector of the ith network element, the dot denotes the time derivative, $\mathbf{F}$ is the nonlinear operator of the network-element evolution, $\mathbf{H}$ is the operator specifying the mutual coupling of elements, $\mathbf{g}_i$ is the vector of control parameters of the ith element, and σ is the coupling parameter. The coupling-coefficient matrix $\mathbf{G}$ is chosen such that all N eigenvalues of the matrix are real and $\lambda_1 \leq \cdots \leq \lambda_N$. In addition, the matrix $\mathbf{G}$ should satisfy the coupling-dissipativity condition $G_{ii} = -\sum_{j \neq i} G_{ij}$ ensuring the possibility of existence of a completely synchronous regime of the network dynamics for which all the network elements show identical behavior, i.e., $\mathbf{x}_i(t) = \mathbf{x}_\mathrm{s}(t)$, where $i = 1, 2, \ldots, N$.

To explain the proposed method, we consider a network comprising identical elements such that $\mathbf{g}_i = \mathbf{g}$, where $i = 1, 2, \ldots, N$. It was mentioned above that for such a network, we have a method allowing us to calculate the range of the coupling parameter σ for which the synchronous state of system (1) is stable [12, 13]:

$$\mathbf{x}_i(t) = \mathbf{x}_\mathrm{s}(t), \qquad i = 1, 2, \ldots, N. \tag{2}$$

For brevity, the regime of complete synchronization (2) of all network elements is called the synchronous state of the network.

The interval of the values of the coupling parameter σ for which the network is in the stable synchronous state can be determined by using N linearized equations for a small deviation of the network state [12, 14]

$$\dot{\boldsymbol{\zeta}}_i = [\mathbf{JF}(\mathbf{x}_\mathrm{s}, \mathbf{g}) - \sigma\lambda_i \mathbf{JH}(\mathbf{x}_\mathrm{s})]\, \boldsymbol{\zeta}_i, \tag{3}$$

where $\mathbf{J}$ is the Jacobian of the matrix $\mathbf{G}$. Note that equations entering system (3) differ only by the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ of the coupling-coefficient matrix $\mathbf{G}$. If we introduce the notation $\sigma\lambda_i = \nu$ in Eq. (3), then the largest Lyapunov exponent Λ [12] totally determines the synchronous-regime stability of the dynamics of network (1). Obviously, the synchronous state $\mathbf{x}_\mathrm{s}(t)$ of the network is stable only if any deviation from the synchronous dynamics decays with time. In other words, all Lyapunov exponents of system (3) should be negative, except for the zero Lyapunov exponent $\lambda_1 = 0$ characterizing the trajectory motion on the synchronous attractor, i.e., $\lambda_i < 0$, where $i = 2, \ldots, N$. Therefore, to analyze stability of the synchronous state $\mathbf{x}_\mathrm{s}(t)$, it is sufficient to consider only one differential equation that depends on the parameter ν and describes the dynamics of a small deviation $\boldsymbol{\zeta}$ with time:

$$\dot{\boldsymbol{\zeta}} = [\mathbf{JF}(\mathbf{x}_\mathrm{s}, \mathbf{g}) - \nu \mathbf{JH}(\mathbf{x}_\mathrm{s})]\, \boldsymbol{\zeta}. \tag{4}$$

Since the interelement interaction in the synchronous state is vanishingly small, the synchronous regime $\mathbf{x}_\mathrm{s}(t)$ is described by the equation

$$\dot{\mathbf{x}}_\mathrm{s}(t) = \mathbf{F}(\mathbf{x}_\mathrm{s}(t), \mathbf{g}). \tag{5}$$

for one autonomous element of the network. Obviously, the synchronous state $\mathbf{x}_\mathrm{s}(t)$ is stable if a small perturbation $\boldsymbol{\zeta}(t)$ decays with time. The behavior of the small perturbation $\boldsymbol{\zeta}(t)$ can be characterized using the largest Lyapunov exponent which can be determined numerically [15].

The considered largest Lyapunov exponent $\Lambda(\nu)$ can be negative either on a finite interval $I_\mathrm{st} = (\nu_1, \nu_2)$ of the values of the parameter ν [12] or an infinite interval (for example, $\nu_2 = \infty$), or on several finite intervals (ν_i, ν_{i+1}), depending on a particular choice of the operators $\mathbf{F}$ and $\mathbf{H}$. In what follows we consider the case where the function $\Lambda(\nu)$ is negative on the interval (ν_1, ν_2), where ν_2 can have both finite and infinite values. Then the synchronous regime of the dynamics of network (1) for a certain value of the coupling parameter σ is stable if the conditions $\sigma\lambda_2 > \nu_1$ and $\sigma\lambda_N < \nu_2$ hold for the chosen value of σ. The operators $\mathbf{F}$ and $\mathbf{H}$ determine the boundary values ν_1 and ν_2 of the stability interval I_st, and the eigenvalues λ_i of the coupling-coefficient matrix $\mathbf{G}$ are determined only by the network topology, i.e., the interelement-coupling structure itself.

We now turn to analyzing the behavior of the networks consisting of the elements whose control parameters are slightly nonidentical. In this case, Eq. (4) for the dynamics of a small deviation is no longer valid, and the synchronous-state stability should be analyzed for initial system (1). Since the control-parameter vector $\mathbf{g}_i$ varies from one element to another, i.e., depends on the element number i, the invariant manifold of the synchronous regime $\mathbf{x}_i(t) = \mathbf{x}_{\mathrm{s}}(t)$, where $i = 1, 2, \ldots, N$, no longer exists for the dynamics of network (1), and, correspondingly, the synchronous regime is not an invariant manifold in the case of nonidentical elements.

However, it is shown in [16] that slight nonidentity of the control parameters of coupled systems can be simulated by introducing identical elements of weak noise to the corresponding system. Then the problem of determining the stability interval of synchronous regime of the network of nonidentical elements can be reduced to analyzing the corresponding system of identical elements with the control-parameter vector $\mathbf{g} = \langle \mathbf{g}_i \rangle$, where the angular brackets denote averaging over all network elements. In this paper, using numerical calculations, we show adequacy of such an approach to analysis of the synchronous regime of a network comprising slightly nonidentical elements.

The proposed method can be demonstrated by an example of the well-known phenomenon [16] observed in two coupled identical chaotic oscillators for which two bifurcation values of the coupling parameter σ exist such that $\sigma_1 < \sigma_2$. The coupling-parameter value σ_1 determines the onset time of the blowout bifurcation when the maximum transversal Lyapunov exponent crosses zero and enters the region of negative values [16], while the coupling-parameter value σ_2 corresponds to loss of tangential stability by the lowest-period periodic orbit built in the synchronous manifold. In the interval $\sigma_1 < \sigma < \sigma_2$ of the coupling-parameter values, the bubbling phenomenon is observed [17]. If two coupled identical oscillators are considered, then for all coupling-parameter values such that $\sigma > \sigma_1$ stable synchronous regime is observed after the transient process. However, if the control parameters of the coupled oscillators slightly differ, then synchronous behavior can be observed only for coupling-parameter values exceeding the threshold σ_1. A similar phenomenon related to the shift of the synchronization threshold to the coupling-parameter value σ_2 is observed for two identical oscillators in the presence of noise, as is shown in [16]. Therefore, in both cases (the case of slight nonidentity of coupled partial subsystems or introduction of weak noise to identical systems), the onset time of the complete-synchronization regime shifts to larger values of the coupling parameter σ and is determined by the value σ_2.

Therefore, the stability interval of the synchronous regime of a network consisting of elements with slightly nonidentical control parameters $\mathbf{g}_i$ can be estimated by analyzing a network of identical elements with the control parameters $\mathbf{g} = \langle \mathbf{g}_i \rangle$ in the presence of a noise source. The method of determining the coupling-parameter variation range in which the synchronous state of the network is stable is reduced to numerical analysis of Eq. (4), where $\mathbf{g} = \langle \mathbf{g}_i \rangle$ is chosen as the control-parameter vector, while Eq. (5) describing the dynamics of the synchronous state $\mathbf{x}_{\mathrm{s}}(t)$ of the considered network should be replaced by the following stochastic differential equation:

$$\dot{\mathbf{x}}_{\mathrm{s}}(t) = \mathbf{F}(\mathbf{x}_{\mathrm{s}}(t)) + D\boldsymbol{\xi}(t). \tag{6}$$

Here $D\boldsymbol{\xi}(t)$ is the introduced noise of intensity D with the variance $\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}(t') \rangle = \delta(t - t')$ and the mean $\langle \boldsymbol{\xi}(t) \rangle = 0$, where $\delta(t)$ is a delta function.

Therefore, when simulating the system of equations (4) and (6), we can also use the above-described calculation of the maximum Lyapunov exponent $\Lambda^D(\nu)$ for finding the interval $I_{\mathrm{st}}^D = (\nu_1^D, \nu_2^D)$ of the values of the parameter ν for which the synchronous state of the network is stable. The noise level D used in Eq. (6) simulates the nonidentity degree of the elements of the considered network. As we show below, the boundaries ν_1^D and ν_2^D of the stability interval I_{st}^D tend to their limiting values ν_1^* and ν_2^*, respectively, with increasing noise intensity D. In the case of a complex network, the limiting values ν_1^* and ν_2^* of the parameter ν are analogs of the above-discussed value σ_2 of the coupling parameter in the case of two coupled chaotic oscillators.

Let us illustrate the proposed method by an example of a network of coupled Rössler systems. The network dynamics is described by Eq. (1), in which $\mathbf{x}_i = (x_i, y_i, z_i)^{\mathrm{T}}$, $\mathbf{g}_i = \omega_i$, $\mathbf{F}(\mathbf{x}_i, \mathbf{g}_i) = \mathbf{F}(\mathbf{x}_i, \omega_i) = (-\omega_i y_i - z_i, \omega_i x_i + 0.165 y_i, 0.2 + z_i(x_i - 10))^{\mathrm{T}}$, and $\mathbf{H}[\mathbf{x}] = (x, 0, 0)^{\mathrm{T}}$, and a Rössler system with the control parameters corresponding to the chaotic regime is chosen as the network element:

$$\dot{x}_i = -\omega_i y_i - z_i - \sigma \sum_{j=1}^{N} G_{ij} x_j, \quad \dot{y}_i = \omega_i x_i + 0.165 y_i, \quad \dot{z}_i = 0.2 + z_i(x_i - 10) \tag{7}$$

The control parameters of the elements (eigenfrequencies ω_i of oscillators) are slightly nonidentical and uniformly distributed near the mean value $\bar{\omega} = \langle \omega_i \rangle = 1$ with the maximum deviation $\Delta\omega \approx 0.1$ from the mean value. When performing numerical calculations, we checked that all elements with slightly nonidentical parameters demonstrate the same behavior type.

Figure 1 shows the dependence of the largest Lyapunov exponent Λ on the parameter ν, which is calculated by the Benettin algorithm [15] on the basis of Eqs. (4) and (5), which simulate the behavior of a network consisting of coupled identical Rössler systems (eigenfrequencies of all the elements of network (7) are identical: $\omega_i \equiv \bar{\omega} = 1$). Figure 1 shows the stability interval $I_{\mathrm{st}} = (a, b)$ determining stable synchronous regime of the dynamics of a network whose all elements are identical.

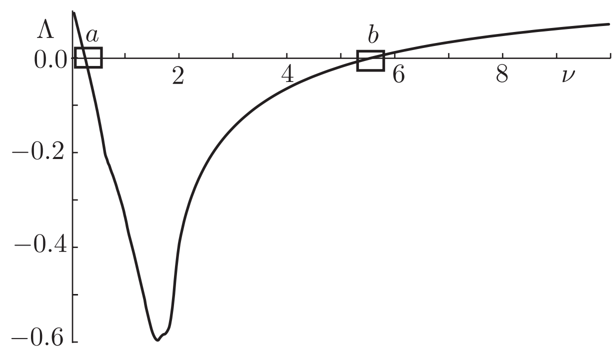


Fig. 1. Dependence of the maximum Lyapunov exponent Λ on the parameter ν for a network of Rössler systems with identical control parameters $\omega_i \equiv \bar{\omega} = 1$. Calculation is performed by numerical simulation of Eqs. (4) and (5). The studied network demonstrates synchronous behavior in the interval $a < \nu < b$. The quantity Λ reverses sign at the points a and b.

To allow for slight nonidentity of the eigenfrequencies of coupled elements of the network, we performed numerical simulation of Eqs. (4) and (6), which allows us to find the stability interval I_{st}^D. To calculate the largest Lyapunov exponent $\Lambda^D(\nu)$ characterizing the synchronous dynamics of such a network, noise simulated by the probabilistic process $\xi(t)$ and uniformly distributed over the interval $(-1, 1)$ was introduced to Eq. (6) for small deviations from the synchronous state. Equation (6) was integrated by the one-step Euler method [18] with the time step $\Delta t = 10^{-6}$.

Figure 2 shows the fragments of the dependence $\Lambda^D(\nu)$ near the threshold values ν_1 and ν_2. It is evident that noise addition to Eq. (6) for small deviations from the synchronous regime results in such a shift of the boundaries ν_1^D and ν_2^D of the stability interval of the synchronous regime that the value of the interval I_{st}^D decreases. Therefore, the synchronous regime for a network of slightly nonidentical elements is stable over a smaller interval of values of the coupling parameter σ compared with a similar network comprising identical elements.

For further analysis of the considered network, we introduce such a characteristic as the relative duration $L_{\mathrm{st}}^D / L_{\mathrm{st}}$ of the stability interval I_{st}^D, where $L_{\mathrm{st}}^D = \nu_2^D - \nu_1^D$ and $L_{\mathrm{st}} = \nu_2 - \nu_1$. Figure 3 shows $L_{\mathrm{st}}^D / L_{\mathrm{st}}$ as a function of the intensity D of noise introduced to Eq. (6) which simulates the synchronous-state behavior of network (7) of slightly nonidentical elements. It is clearly seen that the length of the interval L_{st}^D tends to its limiting length L_{st}^{D*}, which is independent of the value of D, as the noise intensity D increases. By analogy, the boundaries ν_1^D and ν_2^D of the stability interval I_{st}^D converge to the limiting points ν_1^* and ν_2^*, respectively. Therefore, the obtained interval I_{st}^* of the parameter ν yields an estimate of the interval in which a network consisting of nonidentical Rössler systems shows stable synchronous dynamics. It is noteworthy that the obtained interval I_{st}^D is observed for a gradual increase in the noise intensity D up to

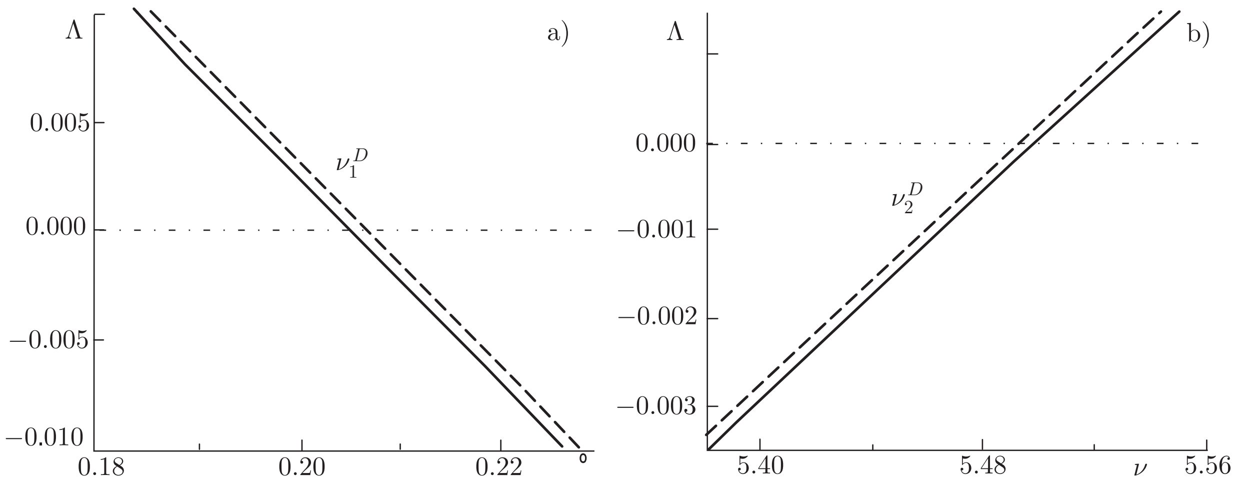


Fig. 2. Fragments of the dependence of the maximum Lyapunov exponent Λ^D on the parameter ν, which is calculated by the above-mentioned method on the basis of Eqs. (4) and (6) for simulating a network of Rössler oscillators with slightly nonidentical parameters ω_i (dashed line). The noise intensity is chosen equal to $D = 3.5$. Similar fragments of the dependence $\Lambda(\nu)$ for a network of identical elements, calculated by Eqs. (4) and (5) (solid line), are also shown. These fragments correspond to the vicinities of the points a and b in Fig. 1.

reasonable values of order $D = 10$–12. At the same time, the proposed method can be applied only for slight nonidentity of the network elements [16] and, correspondingly, for relatively small noise intensities D. For example, if the noise intensity D in Eq. (6) is too large, then the amplitude of the term $D\boldsymbol{\xi}(t)$ is comparable with the signal amplitude $\mathbf{x}_\mathrm{s}(t)$ in the absence of noise, and the network-element dynamics can in fact be destroyed by noise. As a result, the proposed method will yield *a fortiori* false results.

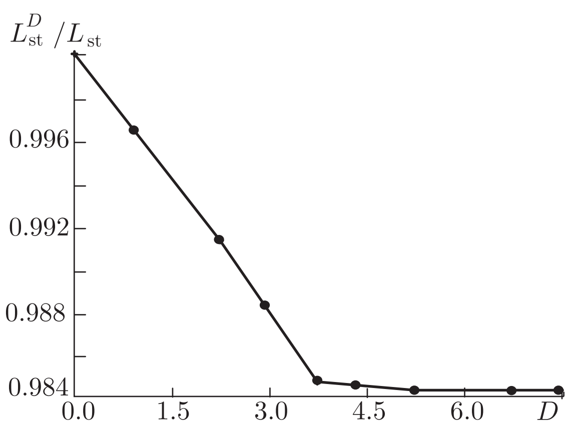


Fig. 3. Dependence of the length L_st^D of the stability interval I_st^D of the synchronous regime on the noise intensity D, which is used in the system of equations (4) and (6) for simulating slight nonidentity of the control parameters ω_i of the network elements. The length L_st^D is normalized to the length L_st of the stability interval I_st of the synchronous regime for a network consisting of identical Rössler systems.

Let us perform numerical simulation of the dynamics of network (7) of nonlinear elements, which allows us to directly estimate the regimes observed in the network and compare the obtained results with the above conclusions drawn on the basis of consideration of Eqs. (4) and (6). In the numerical experiments, we consider network (7) consisting of five elements ($N = 5$). The coupling-coefficient matrix $\mathbf{G}$ is characterized by N real eigenvalues $\lambda_1 = 0.0$, $\lambda_2 \approx -1.6$, $\lambda_3 \approx -2.0$, $\lambda_4 \approx -4.0$, and $\lambda_5 \approx -4.4$ and has the following form:

$$\mathbf{G} = \begin{pmatrix} -2 & 0 & 0 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & -3 \end{pmatrix}.$$

Let us consider the behavior of the synchronization error [14] for the network on the basis of coupled identical elements and for a similar network of elements with nonidentical parameters as the coupling parameter σ varies. The synchronization error is the average (over the time T) deviation from the synchronous state of the network of elements:

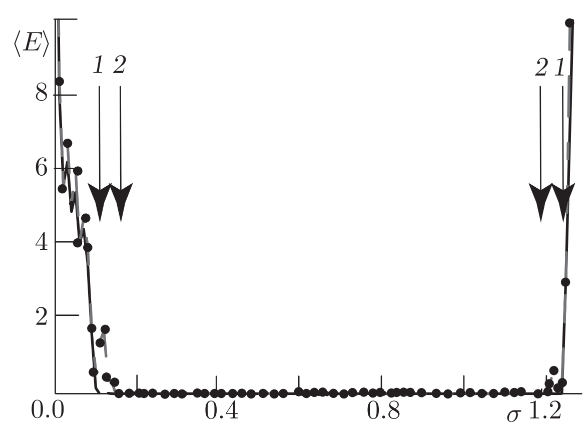
$$\langle E\rangle = \frac{1}{T\,(N-1)} \sum_{j>1} \int\limits_{t}^{t+T} \|\mathbf{x}_j - \mathbf{x}_1\|\, \mathrm{d}t', \tag{8}$$

where the vector norm is calculated as $\|\mathbf{x}\| = |x| + |y| + |z|$. It is evident that the value of $\langle E\rangle$ is close to zero in the regime of complete synchronization, while the synchronization error for the nonsynchronized network differs from zero. Figure 4 shows the results of numerical simulation of a network consisting of identical Rössler systems with the eigenfrequencies $\omega = 1$ and a network whose elements are the Rössler systems with slightly nonidentical control parameters such that the mean value of the eigenfrequency is $\bar{\omega} = 1$ and the maximum deviation from the mean frequency is $\Delta\omega = 0.1$. The range of the coupling parameter σ, in which the studied system (7) demonstrates the regime of complete synchronization, decreases in the case of nonidentical network elements, which is in fairly good agreement with the above analysis of the same network by the proposed method based on Eqs. (4) and (6).

The problem of how the noise intensity in Eq. (6) is related to the spread of control parameters of initial network (7) is very important. To answer this question for the analyzed network of slightly nonidentical coupled elements, we consider a characteristic similar to the relative duration $L_{\mathrm{st}}^{D}/L_{\mathrm{st}}$ of the stability interval I_{st}^{D} whose form is shown in Fig. 3. For network (7) of slightly nonidentical Rössler systems, the stability interval I_{st}^{Δ} corresponding to the stable synchronous dynamics is determined as the interval of the coupling parameter σ for which synchronization error (8) tends to zero. Figure 5 shows the dependence of the relative length $L_{\mathrm{st}}^{\Delta}/L_{\mathrm{st}}$ of the stability interval I_{st}^{Δ} on the maximum deviation $\Delta\omega$. The form of the characteristic corresponds to the dependence of $L_{\mathrm{st}}^{D}/L_{\mathrm{st}}$ on the intensity D of noise used in Eq. (6), which is shown in Fig. 3. Note that L_{st}^{Δ} tends to the limiting value $L_{\mathrm{st}}^{\Delta *}$, which is independent of $\Delta\omega$ and coincides with the corresponding value L_{st}^{D*} in Fig. 3, as the nonidentity of the network elements increases. Therefore, we can state that simulation of the network of coupled nonlinear oscillators with slightly nonidentical control parameters by introducing noise to the equations describing the synchronous-state stability of a network of identical elements yields good correspondence between the introduced noise intensity D and the spread $\Delta\omega$ of control parameters of the elements. Figure 6 shows the relationship between the noise-intensity level D and the maximum deviation $\Delta\omega$, i.e., the nonidentity level of the coupled elements of network (7).

Thus, in this paper, we presented the method allowing us to estimate the interval of the coupling-parameter values for which the network of an arbitrary number of slightly nonidentical elements demonstrates a stable synchronous regime. Note that the proposed results are of general character and can be used for a wide class of dynamical systems such as, for example, relaxation radioengineering oscillators or systems of vacuum microelectronics.

Fig. 4. Synchronization error $\langle E\rangle$ as a function of the coupling parameter σ for the network of Rössler systems with identical control parameters $\omega_i \equiv \omega = 1$ (solid line) and for the network of Rössler systems with slightly nonidentical control parameters $\Delta\omega_i = 0.1$ (broken line and circles). Arrows *1* show the interval of the coupling parameter σ, in which a network of identical oscillators demonstrates a negligibly small value of the synchronization error, and arrows *2* show the interval of σ, in which the synchronization error tends to zero for a network of slightly nonidentical elements. When calculating the synchronization error, we numerically simulated the Rössler-system network described by Eq. (7).



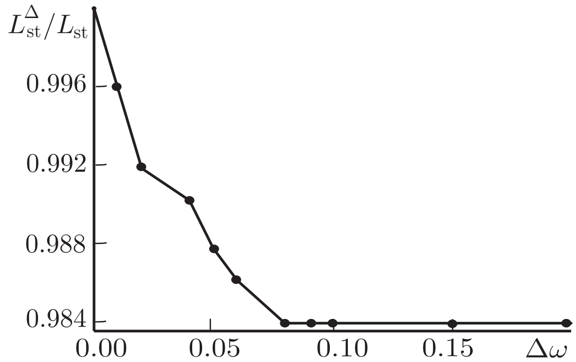


Fig. 5. Length L_{st}^{Δ} of the stability interval I_{st}^{Δ} of the synchronous regime as a function of the value $\Delta\omega$ of nonidentity of the control parameters ω_i for the network elements. The length L_{st}^{Δ} is normalized to the length L_{st} of the stability interval I_{st} of the synchronous regime for a network consisting of identical Rössler systems.

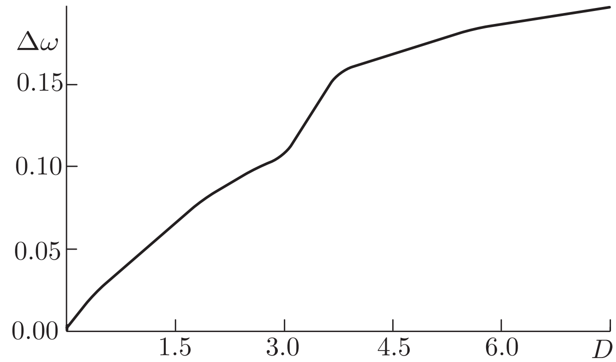


Fig. 6. Correspondence between the intensity D of noise used in the system of equations (4) and (6) and the value $\Delta\omega$ of nonidentity of control parameters for the elements of network (7).

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