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Generalized Synchronization of Chaotic Oscillators

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Abstract—The behavior of two unidirectionally coupled chaotic oscillators near the boundary of their generalized synchronization is analyzed. Using the modified system method, the position of this boundary on the plane of control parameters is considered and physical mechanisms leading to the establishment of the generalized synchronization regime are elucidated.

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The chaotic synchronization of oscillations in dynamical systems is among the basic phenomena extensively studied in recent years [1, 2]. According to the commonly accepted classification, there are several types of synchronous behavior, including phase (PS) [2], generalized (GS) [3], lag (LS) [4], complete (CS) [5], and time-scale (TS) [6] synchronization. Each particular type of synchronous chaotic dynamics has its own distinctive features, and the corresponding methods of diagnostics have been developed, but the question of the relationships between various types of synchronous behavior has been actively discussed in recent years [7–12]).

The aforementioned GS regime is an interesting and important type of the synchronous behavior of a system of unidirectionally coupled chaotic oscillators (subsystems). According to the definition of the GS regime [3], the states of the interacting drive ($x_d(t)$) and response ($x_r(t)$) chaotic subsystems occurring in this regime upon termination of the transient process are related as $x_r(t) = \mathbf{F}[x_d(t)]$, where $\mathbf{F}$ is a certain function. There are several approaches to the diagnostics of GS between the drive and response subsystems, one convenient procedure being offered by the auxiliary system method [13].

It has been reported [14] that the coupling parameter ε_{GS}, for which two coupled chaotic subsystems exhibit GS, is approximately two times greater for small values of the mismatch between the drive and response subsystems than for a large mismatch. For the chaotic synchronization regimes of all other types, the dependence of the synchronization threshold on the mismatch parameter is different: the threshold coupling parameter decreases with the decreasing mismatch. The nature of the anomalous behavior of the GS threshold remained unclear thus far.

The mechanism of the GS establishment can be elucidated using the modified system method [15]. Let us consider the behavior of two unidirectionally coupled chaotic oscillators with slightly different parameters:

$$\dot{\mathbf{x}}_d = \mathbf{H}(x_d, g_d), \quad \dot{\mathbf{x}}_r = \mathbf{H}(x_r, g_r) + \varepsilon\mathbf{A}(x_d - x_r), \quad (1)$$

where $x_{d,\,r}$ are the state vectors of the drive and response subsystems, respectively; $\mathbf{H}$ is the vector field of the system under consideration; g_d and g_r are the vectors of parameters; $\mathbf{A} = \{\delta_{ij}\}$ is the coupling matrix ($\delta_{ii} = 0$ or 1; $\delta_{ij} = 0$ for $i \neq j$); and ε is the coupling parameter. In this case, the response subsystem $x_r(t)$ can be considered as a nonautonomous modified system occurring under the external action $\varepsilon\mathbf{A}\mathbf{x}_d(t)$:

$$\begin{aligned} \dot{\mathbf{x}}_m &= \mathbf{H}'(x_m, g_r, \varepsilon) + \varepsilon\mathbf{A}\mathbf{x}_d, \\ \mathbf{H}'(x, g) &= \mathbf{H}(x, g) - \varepsilon\mathbf{A}\mathbf{x}, \end{aligned} \quad (2)$$

where the term $-\varepsilon\mathbf{Ax}$ in fact introduces an additional dissipation into the modified system. Evidently, the GS regime established in system (1) upon an increase in the coupling parameter (ε) can be considered as a result of two simultaneous and mutually related processes: (i) an increase in the level of dissipation in the modified system and (ii) an increase in the amplitude of the external signal. These processes are interrelated via the parameter ε and cannot proceed in the response subsystem (1) independently. Nevertheless, separate analysis of these processes provides a better understanding of the mechanisms leading to the establishment of the GS regime in the system studied.

In an autonomous modified system, the ε value plays the role of a dissipation parameter. For $\varepsilon = 0$, the behavior of the modified system $x_m(t)$ coincides with that of the response system $x_r(t)$ in the absence of coupling. As the ε value increases, the dynamics of the modified system must simplify. This is manifested by the passage from chaotic oscillations to regular (peri-

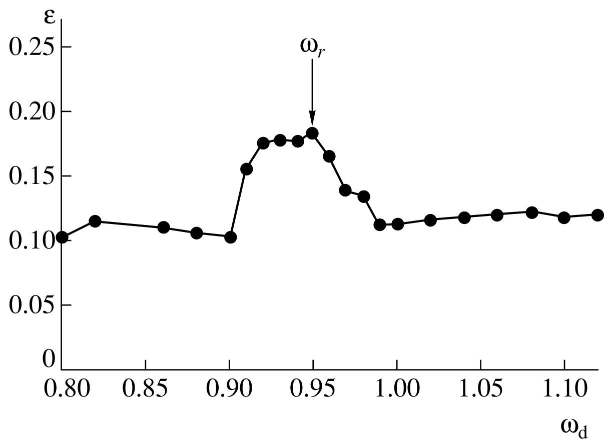

Fig. 1. The boundary of the region corresponding to the onset of the GS regime on the plane of control parameters (ω_d, ε) for a system of two unidirectionally coupled Rössler oscillators. The arrow indicates the ω_r value of the response system.

odic) oscillations or to a stationary state (at large ε values). On the contrary, the external signal in relation (2) tends to induce the chaotic dynamics $x_d(t)$ of the drive system to the modified system $x_m(t)$, thus complicating its dynamics.

Recently, we have demonstrated [15] that the GS regime in system (1) can exist only provided that the intrinsic chaotic dynamics of the modified system (2) is suppressed as a result of the increased dissipation. However, it is well known that even a periodic external action can lead to the development of chaotic dynamics in a system, which initially exhibited periodic oscillations (see, e.g., [16]). Therefore, the established regular regime must be sufficiently stable, so that the external action would not excite the intrinsic chaotic dynamics of the modified system $x_m(t)$. The stability of a periodic regime is determined primarily by the properties of the modified system and is independent of the mismatch of parameters. Therefore, in the study of mechanisms accounting for the appearance of a GS regime, it is expedient to fix the control parameters g_r of the response system and vary the parameters g_d of the drive oscillator.

Now let us consider the threshold of the GS regime in a system comprising two one-way coupled chaotic oscillators representing Rössler systems with slightly mismatched parameters:

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, \quad \dot{x}_r = -\omega_r y_r - z_r + \varepsilon(x_d - x_r), \\ \dot{y}_d &= \omega_d x_d + a y_d, \quad \dot{y}_r = \omega_r x_r + a y_r, \qquad (3) \\ \dot{z}_d &= p + z_d(x_d - c), \quad \dot{z}_r = p + z_r(x_r - c), \end{aligned}$$

where ε is the coupling parameter and subscripts “d” and “r” refer to the drive and response subsystems, respectively. The values of the control parameters are selected by analogy with those used in [14]: $a = 0.15$, $p = 0.2$, and $c = 10.0$. The control parameter of the driven (response) system, $\omega_r = 0.95$ (which characterizes its fundamental frequency) is fixed. The analogous parameter of the drive system, ω_d, is varied in the interval from 0.8 to 1.1, which corresponds to a slight mismatch of the two oscillators.

Figure 1 shows the boundary of the region corresponding to the onset of the GS regime on the plane of control parameters (ω_d, ε) for the system under consideration. The PS threshold was determined by calculating the effective Lyapunov exponents for system (3) and then refined using the auxiliary system method. As can be seen, the threshold for the onset of the PS regime is much lower for a small mismatch of the control parameters of subsystems than for the large deviations. On the other hand, it should be also noted that the coupling parameter ε_{GS} corresponding to the GS onset at a large mismatch is virtually independent of the ω_d value.

The observed behavior of the coupled subsystems can be explained using the modified system method, which was briefly outlined above (see also [15]). Indeed, the response subsystem in (3) can be reduced to an autonomous modified system

$$\begin{aligned} \dot{x}_m &= -\omega_r y_m - z_m - \varepsilon^* x_m, \\ \dot{y}_m &= \omega_r x_m + a y_m, \qquad (4) \\ \dot{z}_m &= p + z_m(x_m - c), \end{aligned}$$

where the coefficient ε* is equal to the coupling parameter ε, while the asterisk is used in order to distinguish between the two mechanisms (variable dissipation and the external action intensity) accounting for the GS onset. As the ε* value is increased, the modified Rössler system passes from the chaotic to periodic oscillations via a cascade of period doubling bifurcations.

In order to study the GS regime, let us consider the nonautonomous dynamics of the modified system

$$\begin{aligned} \dot{x}_m &= -\omega_r y_m - z_m - \varepsilon^* x_m + \varepsilon F(t), \\ \dot{y}_m &= \omega_r x_m + a y_m, \\ \dot{z}_m &= p + z_m(x_m - c), \end{aligned}$$

where the external signal $F(t)$ is represented by a time series $x_d(t)$ of the drive oscillator of system (3). At a sufficiently large value of ε*, the autonomous modified system will occur in a periodic regime. At the same time, the Fourier spectrum of the drive subsystem in (3) at the selected values of control parameters exhibits a pronounced spectral component corresponding to the fundamental frequency. Accordingly, the first approximation in the description of unidirectionally coupled oscillators (3) is offered by the behavior of the above modified system under the action of the external harmonic drive $F(t) = A\cos(\Omega t)$, where the amplitude A and the frequency Ω are those of the fundamental component of the driving Rössler oscillator.

It is evident that, for the Ω values close to the fundamental frequency of the modified system (4), the frequency entrainment can take place and, hence, the synchronization is possible. Therefore, the behavior of the modified system (and of the response system) inside and outside the GS domain can be qualitatively different. This assumption is confirmed by the position of the GS onset boundary in Fig. 1. The fact that the GS at a large mismatch of the parameters of subsystems is established virtually at the same coupling parameter $\varepsilon_{GS} \approx 0.11$ (irrespective of ω_d) indicates that the subsystems are outside the synchronization domain (no entrainment of the fundamental frequency). In this state, the ε_{GS} value is determined primarily by the properties of the modified system (4). The position of the boundary of the GS onset at a small mismatch of the control parameters is determined by the system dynamics inside the synchronization "tong" on the plane of parameters.

In order to elucidate the factors influencing the position of the GS boundary on the (ω, ε) plane, we have analyzed the nonautonomous behavior of the modified system (4) under the external harmonic action. The parameter ε^* was fixed at 0.11 (which approximately corresponds to the boundary of the GS onset at a large mismatch of the coupled oscillators) and the external action amplitude was taken in the form f $A = \varepsilon B$, where $B \approx 0.105$ is selected so that the harmonic signal energy would correspond to the energy of the fundamental frequency component of the drive subsystem. In this case, the parameter ε^* determines the intensity of external action and is analogous to the coupling parameter in the initial system (3). Using this normalization, it is possible to determine the boundaries of the synchronization region for the nonautonomous modified system on the (ω_d, ε) plane, since the drive frequency Ω has the meaning of the fundamental frequency in the Fourier spectrum of the drive Rössler subsystem, which is very close to ω_d.

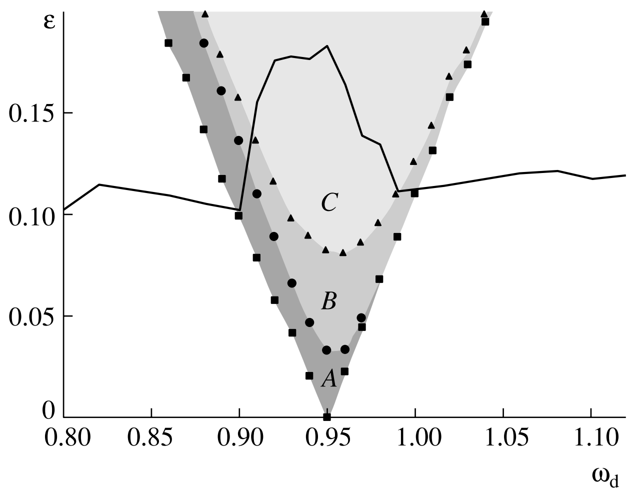

Fig. 2. The plane of control parameters (ω_d, ε) showing the synchronization "tong" for the nonautonomous modified system under the external harmonic action, where region A corresponds to the periodic oscillations of the cycle of period 1, region B corresponds to the cascade of period doubling bifurcations, and region C, to the chaotic oscillations. Solid curve shows the boundary of the onset of GS in the initial system, which is plotted on the (ω_d, ε) plane in order to explain the anomalous behavior of the GS threshold in the region of small mismatch of parameters of the drive and response subsystems.

Figure 2 shows the synchronization "beak" on the $(\omega_d = \Omega, \varepsilon)$ plane for the modified Rössler system (4) under the external harmonic action with the parameters indicated above. For the comparison, this figure also shows the boundary of the onset of GS in the initial system. A comparison of these patterns leads to a conclusion that the position of the boundary of the GS onset is related to the existence of the synchronization beak. Indeed, outside the beak, the GS onset is determined by the properties of the modified system and is independent of the mismatch of parameters of the drive and response subsystems. Inside the beak, the behavior is qualitatively different. In order to explain why does the boundary of the GS onset at a small mismatch shifts toward greater values of the coupling parameter ε, we depicted the line of period doubling and the line of chaos (generated in the nonautonomous modified system via a cascade of period doubling bifurcations) inside the beak. As can be seen, the system features chaotic oscillations in the region of a small mismatch, this chaos hindering the passage to the GS regime. The generation of chaotic oscillations can be suppressed only at the expense of increasing dissipation in the modified system, which corresponds to an increase in the coupling parameter ε in the initial system (3). Accordingly, the GS threshold in the region of small mismatch shifts toward greater values of the coupling parameter.

When comparing the boundary of the GS onset on the (ω_d, ε) plane to the synchronization "tong" of the nonautonomous modified system, one should bear in mind that the synchronization region on the plane of parameters provides only a qualitative pattern that explains the observed features in behavior of the coupled subsystems. Indeed, the nonautonomous modified system (5) quantitatively coincides with the response subsystem in (3) only at $\varepsilon = \varepsilon^*$, while only qualitative correspondence takes place at all other values of the coupling parameter. In addition, modeling of the external action by a harmonic signal excludes from consideration the chaotic dynamics of the drive subsystem that also influences the process of chaotization in the response subsystem by shifting the boundary of chaos excitation toward lower values of the coupling parameter ε.

To summarize, the position of the boundary of GS on the plane of parameters is determined by the process of entrainment of the fundamental frequency of the drive subsystem by the response subsystem. Inside the synchronization beak, the boundary of the GS onset can shift toward greater values of the coupling parameters when the external action inside the beak excites the intrinsic chaotic dynamics of the modified system.

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REFERENCES

1. A. Grossman and J. Morlet, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **15**, 273 (1984).
2. A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Univ. Press, Cambridge, 2001).
3. N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
4. M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
5. L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).
6. A. E. Hramov and A. A. Koronovskii, Physica D **206**, 252 (2005). 1
7. S. M. Boccaletti, L. M. Pecora, and A. Pelaez, Phys. Rev. E **63**, 066219 (2001).
8. R. Brown and L. Kocarev, Chaos **10**, 344 (2000).
9. A. A. Koronovskiĭ and A. E. Hramov, Pis'ma Zh. Éksp. Teor. Fiz. **79**, 391 (2004) [JETP Lett. **79**, 316 (2004)].
10. A. E. Hramov and A. A. Koronovskii, Chaos **14**, 603 (2004). 1
11. A. A. Koronovskii, O. I. Moskalenko, and A. I. Hramov, Pis'ma Zh. Éksp. Teor. Fiz. **80**, 25 (2004) [JETP Lett. **80**, 20 (2004)]. 1
12. A. E. Hramov, A. A. Koronovskii, M. K. Kurovskaya, and O. I. Moskalenko, Phys. Rev. E **71**, 056620 (2005). 1
13. H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, Phys. Rev. E **53**, 4528 (1996).
14. Z. Zheng and G. Hu, Phys. Rev. E **62**, 7882 (2000).
15. A. E. Hramov and A. A. Koronovskii, Phys. Rev. E **71**, 067201 (2005). 1
16. S. P. Kuznetsov, *Dynamical Chaos* (Fizmatlit, Moscow, 2001) [in Russian].

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SPELL: 1. Koronovskii—ok?