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PHYSICS

Generalized Synchronization and Noise-Induced Synchronization: The Same Type of Behavior of Coupled Chaotic Systems

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In this paper, we demonstrate that two types of synchronization in nonlinear chaotic systems, namely, generalized synchronization and noise-induced synchronization, represent the same type of synchronous behavior, which is due only to the reason of introducing an additional dissipation into the system.

At present, the study of peculiarities of synchronization of coupled chaotic oscillators seems to be a substantial and topical direction of theoretical and experimental research, which is of particular practical importance in analysis and diagnostics of medical and physiological data, information security, and control of radiophysical and laser systems [1–3]. Currently, different types of chaotic synchronization are distinguished; each of them has its own features and methods for detection [1]. It is also important to find and analyze common regularities of chaotic synchronization, which requires establishment of the relationship between its different types. In particular, the results presented in [4, 5] demonstrate that different types of chaotic synchronization in flow systems with a few degrees of freedom may be reduced to time scale synchronization.

In this paper, we show that two types of synchronization of chaotic oscillators (generalized synchronization [6, 7] and noise-induced synchronization [8–10]), which have previously been regarded as different, are due to the same reason and represent the same type of synchronous behavior.

The **generalized synchronization** of unidirectionally coupled chaotic oscillators means the type of synchronous behavior such that the state vectors $\mathbf{x}(t)$ and $\mathbf{u}(t)$ of chaotic oscillators with continuous or discrete time, respectively,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{H}(\mathbf{x}(t)), \\ \dot{\mathbf{u}}(t) &= \mathbf{G}(\mathbf{u}(t)) + \mathbf{E}(\mathbf{u}(t), \mathbf{x}(t)) \end{aligned} \quad (1)$$

and

$$\begin{aligned} \mathbf{x}(n+1) &= \mathbf{H}(\mathbf{x}(n)), \\ \mathbf{u}(n+1) &= \mathbf{G}(\mathbf{u}(n)) + \mathbf{E}(\mathbf{u}(n), \mathbf{x}(n)) \end{aligned} \quad (2)$$

(here, $\mathbf{H}$ and $\mathbf{G}$ are the operators of evolution of the drive and response systems, respectively, and $\mathbf{E}$ is the vector function describing the relation between the drive $\mathbf{x}(t)$ and response $\mathbf{u}(t)$ systems) are related by some functional dependence $\mathbf{F}[\cdot]$ so that $\mathbf{u}(t) = \mathbf{F}[\mathbf{x}(t)]$ [6, 7]. This dependence may be rather intricate and difficult to find. Depending on the shape of the functional dependence—smooth or fractal—the generalized synchronization is classified as strong or weak [7]. As the interacting oscillators, two different dynamical systems (even with different dimensions of the phase space) may be used.

The regime of generalized synchronization between the chaotic oscillators (1) may be detected with the use of the method of auxiliary system [6]: along with the response system $\mathbf{u}(t)$, an identical auxiliary system $\mathbf{v}(t)$ is considered. The dynamics of the latter system is also described by relation (1) with the difference that the state vector $\mathbf{u}(t)$ is replaced by vector $\mathbf{v}(t)$ corresponding to the auxiliary system. The initial state $\mathbf{v}(t_0)$ of the auxiliary system is different from that of the response system $\mathbf{u}(t_0)$, although they both belong to the same basin of chaotic attractors. In the absence of generalized synchronization, the state vectors of the response $\mathbf{u}(t)$ and auxiliary $\mathbf{v}(t)$ systems are different. In the case of the regime of generalized synchronization, by virtue of relations $\mathbf{u}(t) = \mathbf{F}[\mathbf{x}(t)]$ and, accordingly, $\mathbf{v}(t) = \mathbf{F}[\mathbf{x}(t)]$,

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after the transient process is over, the states of the response and auxiliary systems must be identical, i.e., $\mathbf{u}(t) = \mathbf{v}(t)$. Thus, the equivalence of states of the response and auxiliary systems is a criterion for the presence of generalized synchronization between the drive and response oscillators.

The regime of generalized synchronization may also be detected through calculation of the conditional Lyapunov exponents [7]. In this case, the Lyapunov exponents for the response system are calculated. Since the behavior of the response system differs from that of the drive system, these exponents are different from the Lyapunov exponents for the autonomous system and are called conditional. Generalized synchronization is present only if the highest conditional Lyapunov exponent is negative [7].

The regime of **noise-induced synchronization** [8–10] means that a random signal $\xi(t)$ driving two independent (though identical) chaotic systems $\mathbf{u}(t)$ and $\mathbf{v}(t)$ with different initial conditions $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ may cause synchronization of these systems with each other; i.e., after the transient process is over, we have $\mathbf{u}(t) \equiv \mathbf{v}(t)$. In this case, similar to the regime of generalized synchronization, two systems with a common source of noise can have synchronized dynamics only if all the conditional Lyapunov exponents are negative [11].

It was demonstrated earlier that there are two similar mechanisms of establishing the regime of noise-induced synchronization.

(1) The random signal $\xi(t)$ that drives the identical chaotic systems has a zero mean value; this actually brings the system into the nonchaotic regime [12, 13], in which the state of the system just follows the external random noise $\xi(t)$.

(2) The external random signal with rather large amplitude moves the image point into the regions of the phase space with a strong contraction of the phase flow; the image point stays in these regions for most of the time and, thereby, the neighboring trajectories converge on the average [10, 14].

In both cases, the contraction of the phase flow plays the key role and the conditional Lyapunov exponents are negative.

The similar behavior of two identical chaotic systems driven by the same chaotic signal suggests that there is a functional dependence between the state of the dynamical system and the external random signal. Indeed, let us denote the states of the systems by $\mathbf{x}_1$ and $\mathbf{x}_2$. Then, the occurrence of noise-induced synchronization means that $\mathbf{x}_1 = \mathbf{x}_2$. Obviously, $\mathbf{x}_1(t) = \mathbf{F}_1[\xi(t)]$ and $\mathbf{x}_2(t) = \mathbf{F}_2[\xi(t)]$, where $\mathbf{F}_1$ and $\mathbf{F}_2$ are some functional dependences, which are, in general, distinct for different initial conditions $\mathbf{x}_0$. However, the fact that noise-induced synchronization appears implies that the functional dependences $\mathbf{F}_1$ and $\mathbf{F}_2$ coincide and are independent of the initial conditions; i.e., there occurs a unique functional dependence between the state of the dynamical system and the external noise signal.

As is shown in our paper [15], similar effects caused by the introduction of the additional dissipation into the system lead to the establishment of the regime of generalized synchronization. First, the regime of generalized synchronization can be observed for two chaotic oscillators coupled by a unidirectional dissipative coupling with slightly detuned control parameters. In this case, the equations describing the behavior of system (1) may be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{H}(\mathbf{x}(t)), \\ \dot{\mathbf{u}}(t) &= \mathbf{H}(\mathbf{u}(t)) + \varepsilon\mathbf{A}(\mathbf{x}(t) - \mathbf{u}(t)), \end{aligned} \tag{3}$$

where $\mathbf{A} = \{\delta_{ij}\}$ is the coupling matrix, ε is the coupling strength, $\delta_{ii} = 0$ or 1, and $\delta_{ij} = 0$ $(i \neq j)$. It is clear that the response system $\mathbf{u}(t)$ may be considered as a modified system

$$\dot{\mathbf{u}}_m(t) = \mathbf{H}'(\mathbf{u}_m(t), \varepsilon) \tag{4}$$

subjected to the external force $\varepsilon\mathbf{A}\mathbf{x}(t)$:

$$\dot{\mathbf{u}}_m(t) = \mathbf{H}'(\mathbf{u}_m(t), \varepsilon) + \varepsilon\mathbf{A}\mathbf{x}(t). \tag{5}$$

Here, $\mathbf{H}'(\mathbf{u}(t)) = \mathbf{H}(\mathbf{u}(t)) - \varepsilon\mathbf{A}\mathbf{u}(t)$. The term $-\varepsilon\mathbf{A}\mathbf{u}(t)$ introduces an additional dissipation into the modified system (4). All the foregoing is also valid for systems with discrete time (2).

The regime of generalized synchronization arising in system (3) may be considered as a result of two simultaneous cooperative processes, namely, the growth of dissipation in the modified system (4) and the rise of the amplitude of the external signal. The processes are interrelated by means of parameter ε and cannot be realized in the response system (3) independently. However, the growth of dissipation in the modified system (4) results in the simplification of its behavior; the chaotic oscillations are changed to periodic oscillations (or to a stationary state). In contrast, the external force tends to complicate the behavior of the modified system and to impose its own dynamics on it. Obviously, the regime of generalized synchronization cannot arise until the characteristic chaotic dynamics of the response system is suppressed owing to the dissipation.

Second, the drive and response systems may be coupled in a nondissipative way. In this case, the external signal of the drive system should have a rather large amplitude, so that, similar to the case of noise-induced synchronization, it moves the image point into the region of the phase space with a strong contraction of the phase flow, where the neighboring trajectories converge on the average. In both cases, the contraction of the phase flow plays the key role, the conditional Lyapunov exponents are negative, and the regime of

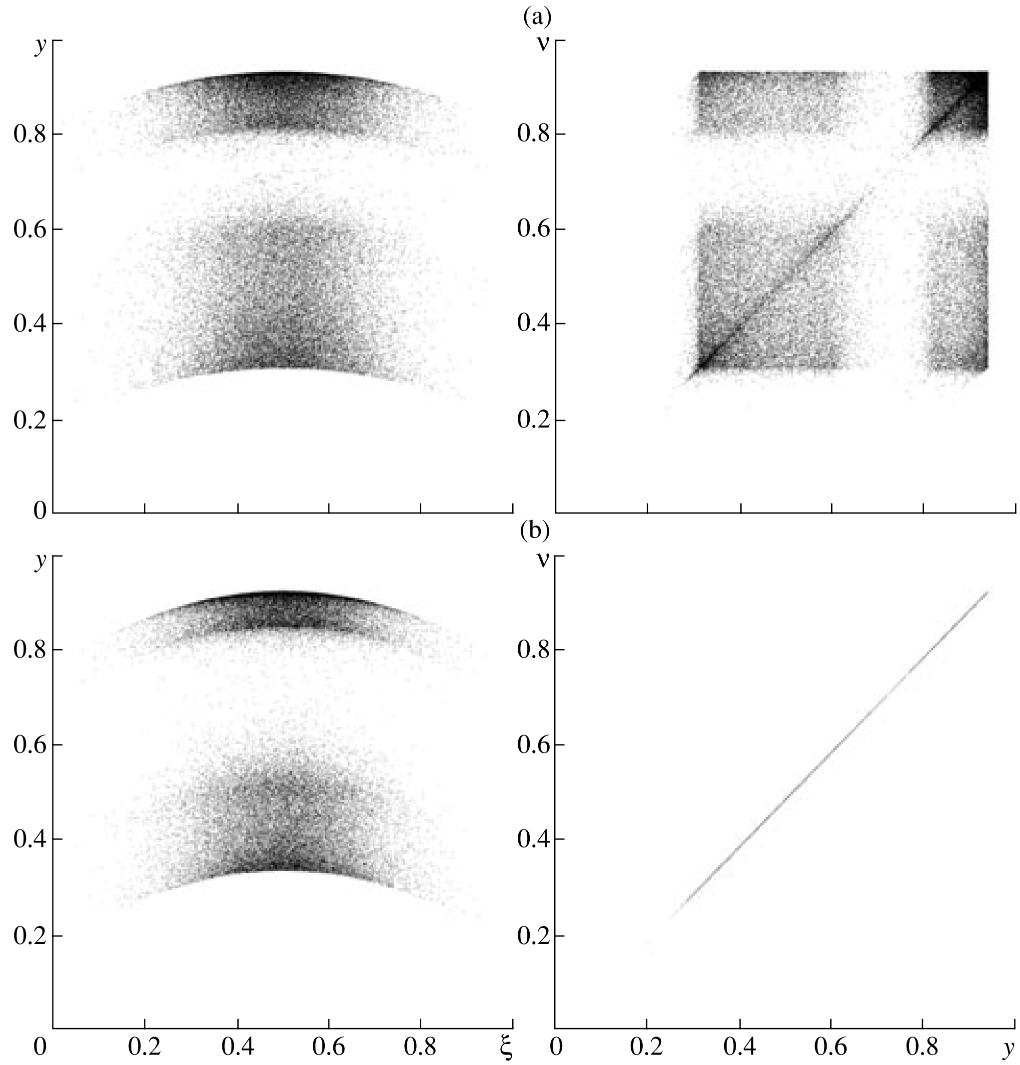

Fig. 1. Planes (ξ_n, y_n) and (y_n, v_n) for the values of the coupling parameter $\varepsilon = 0.125$ (a) and 0.175 (b). For $\varepsilon = 0.175$, the response y_n and auxiliary v_n systems are seen to exhibit the same behavior $y_n = v_n$, which is indicative of the presence of a functional relationship $y_n = F(\xi_n)$ and, accordingly, of the establishment of a synchronous regime.

generalized synchronization is also realized in the system. The latter was illustrated in [10, 15] with the examples of the Rössler and Lorenz coupled systems.

Thus, noise-induced synchronization and generalized synchronization arise from the same mechanism, which indicates the unified character of these two types of synchronization of chaotic oscillators.

Let us demonstrate that the regime of generalized synchronization can be obtained with a random external signal taken as the signal driving the response system. This may also be treated as noise-induced synchronization.

As a model, we consider the system of two unidirectionally coupled dynamical systems with discrete time (logistic maps). It has been shown in [7] that, for such systems, the regime of generalized synchronization is present. The model has the form

$$x_{n+1} = f(x_n), \quad y_{n+1} = f(y_n) + \varepsilon(f(x_n) - f(y_n)), \tag{6}$$

where $f(x) = ax(1-x)$, a is the control parameter, and ε is the coupling parameter. Now, we consider the case where the time behavior of the value of x is not determined by the evolution operator $f(x)$ in (6), but is a random process ξ_n with probability density $p(\xi)$. Then, the dynamics of the response system is described by the equation

$$y_{n+1} = f(y_n) + \varepsilon(f(\xi_n) - f(y_n)). \tag{7}$$

Let us show that, notwithstanding the random character of value ξ, again, this stochastic process and the dynamical system may come to a synchronous behavior simi-

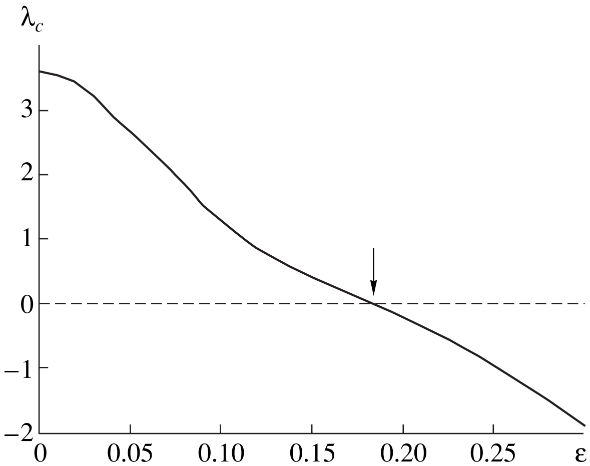

Fig. 2. The conditional Lyapunov exponent λ_c of system (7) versus the coupling parameter ε. The random value ξ_n is normally distributed; the onset of the synchronization is marked by the arrow.

lar to generalized synchronization or noise-induced synchronization.

To detect the synchronization between the random process ξ_n and the dynamical system y_n, let us use the method of auxiliary system. Figure 1a shows the behavior of the response and auxiliary systems (y_n and v_n, respectively) for $a = 3.75$ and normally distributed random value ξ. One can see that, in the case of a small value of the coupling parameter ($\varepsilon = 0.125$), the response and auxiliary systems take different values at the same discrete time instant. Hence, there is no functional dependence between the random process ξ_n and the state of the dynamical system y_n. With increasing value of the coupling parameter ($\varepsilon = 0.175$), things change radically (see Fig. 1b); namely, the points that correspond to the states of the systems lie on the diagonal $y = v$, which is indicative of a functional relationship $y_n = F(\xi_n)$. However, in this case, the functional dependence $\mathbf{F}[\cdot]$ has a complicated fractal structure, which corresponds to the case of a weak synchronization. The presence of the functional relationship cannot be deduced from the analysis of the plane (ξ, y) (compare Fig. 1a, left part).

The presence of the regime of generalized synchronization (or noise-induced synchronization) is also confirmed by the dependence of the conditional Lyapunov exponent λ_c on the coupling parameter ε (Fig. 2). One can see that, for small values of the coupling parameter, λ_c is positive; this is to say that there is no functional dependence between the random value ξ_n and the state of the dynamical system y_n. With increasing value of the coupling parameter, it becomes negative and, hence, the functional dependence $y_n = \mathbf{F}[\xi_n]$ arises, which corresponds to the establishment of the regime of generalized synchronization (or noise-induced synchronization).

Similar results have also been obtained for flow systems (unidirectionally coupled Rössler systems); the corresponding data are not presented here for reasons of economy of space.

Thus, the regimes of generalized chaotic synchronization and noise-induced synchronization, which have traditionally been regarded as different phenomena, are actually due to the same reason, namely, the suppression of characteristic chaotic oscillations by means of introducing additional dissipation.

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