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Time Shift between Unstable Periodic Orbits of Coupled Chaotic Oscillators

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Abstract—The shift Δt between unstable periodic orbits of coupled oscillators occurring in the chaotic synchronization regime has been studied. It is shown that this time shift is the same for all equiphase orbits with various topological parameters and depends on the coupling parameter ε. This dependence obeys the universal power law $\Delta t \sim \varepsilon^n$ with an exponent of $n = -1$. *© 2005 Pleiades Publishing, Inc.*

Chaotic synchronization of dynamical systems is among the important basic phenomena extensively studied in recent years [1]. This process is also of considerable practical interest (e.g., for data transfer by means of deterministic chaotic oscillations [2], for solving some problems in biology [3], etc.). According to modern classification, there are several types of chaotic synchronization of coupled oscillators, including generalized, phase, lag, and complete synchronization (see, e.g., [4]). Recently, it has been shown [5, 6] that the phase, generalized, lag, and complete synchronization regimes are closely related, since they are essentially manifestations of the same type of synchronous dynamics referred to as time scale synchronization. The character of a particular synchronous regime (phase, lag, or complete) is determined by the number of synchronized time scales introduced by means of a continuous wavelet transform [7].

In this Letter, we will analyze how the time shift between synchronized unstable saddle periodic orbits (corresponding to the equiphase saddle m:m cycles in the general phase space) changes in the phase spaces of interacting chaotic oscillators depending on the parameter of coupling between the interacting subsystems.

Unstable saddle periodic orbits [8] play an important role in the process of chaotic synchronization (see, e.g., [9–11]). An autonomous chaotic oscillator is characterized by a set of unstable saddle periodic orbits of various periods, which are incorporated into the chaotic attractor. For a small coupling parameter, each of the two mutually coupled chaotic oscillators with slightly different parameters is characterized by its own set of unstable saddle orbits. The saddle orbits with the same topological period m have different temporal periods T (the time required for the imaging point to return to a fixed point of the orbit) and the corresponding different frequencies. Accordingly, unstable two-dimensional toruses exist in the complete phase space formed by the partial phase spaces of the interacting oscillators.

The onset of phase synchronization is accompanied by trapping of the frequencies of unstable periodic orbits (for more detail, see [12]) and by the appearance of resonance saddle cycles (either equiphase or not) on the two-dimensional toruses. It was shown [12] that only equiphase resonance saddle m:n cycles (where m, $n = 1, 2, \ldots$) exist in a broad range of variation of the coupling parameter, including the regions of phase synchronization and lag synchronization. All other resonance saddle cycles (including nonequiphase m:m cycles) exist in a relatively small interval of the coupling parameter and break when this parameter approaches a threshold of the lag synchronization.

There is still an open question concerning the behavior of a time shift between saddle orbits of the partial systems forming a resonance cycle in the general phase space. The interest in answering this question has been inspired by the results obtained recently [13] in the study of a phase shift between synchronized spectral components of interacting chaotic oscillators.

In the regime of lag synchronization, whereby the state vector of one of the two coupled systems has a certain time lag relative to the state vector of another system, $\mathbf{x}_1(t) \simeq \mathbf{x}_2(t + \Delta t)$, the time shift between the saddle cycles with topological periods m in the coupled systems (corresponding to an equiphase resonance m:m cycle in the general phase space) also proves to be Δt. However, it is still unclear how this time shift behaves in the regime of phased synchronization.

In order to analyze this problem, let us consider two mutually coupled Rössler systems occurring in the

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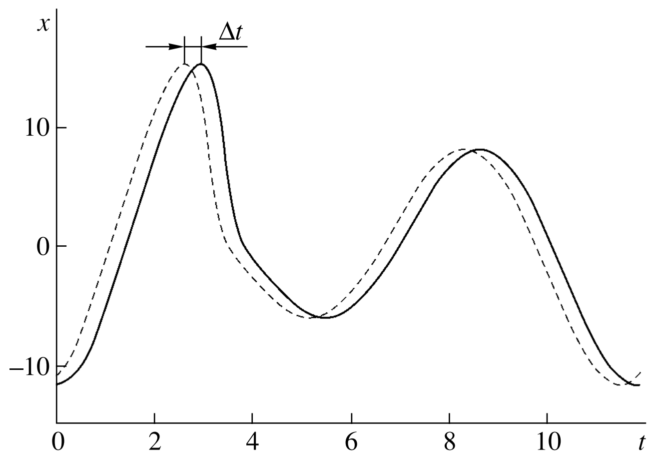

Fig. 1. Time series corresponding to an equiphase resonance saddle m:m cycle with a topological period of $m = 2$ for a coupling parameter of $\varepsilon = 0.07$ corresponding to a phase synchronization regime. The solid and dashed curves show the saddle orbits in the first and second system, respectively.

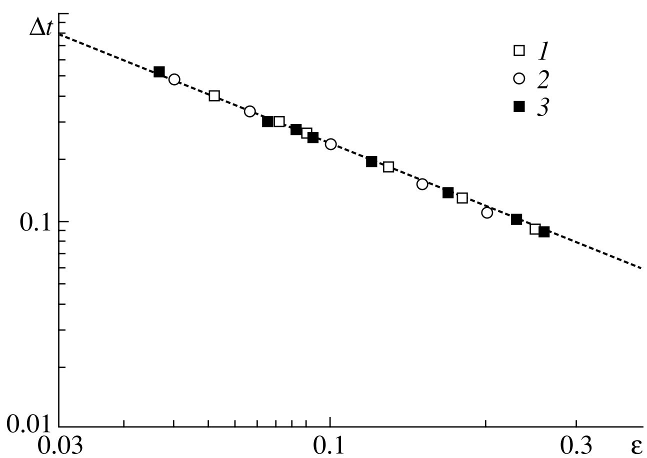

Fig. 2. A double logarithmic plot of the time shift Δt between synchronized saddle periodic orbits versus the coupling parameter ε of coupled Rössler systems (1), for the orbits with the topological periods $m = 1$ (*1*), 2 (*2*), and 3 (*3*). The dashed line corresponds to power law (2) with an exponent of $n = -1$.

dynamical chaos regime:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{1}$$

where ε is the coupling parameter, $\omega_1 = 0.98$, and $\omega_2 = 1.03$. The values of other control parameters were selected as follows: $a = 0.22$, $p = 0.1$, and $c = 8.5$. It is known [14] that two coupled Rössler systems (1) with $0.04 \lesssim \varepsilon \lesssim 0.14$ occur in the regime of phase synchronization; for $\varepsilon > 0.14$, the same systems exhibit lag synchronization.

We have considered equiphase unstable saddle periodic orbits with topological periods, incorporated into chaotic attractors of the interacting chaotic oscillators. The system of equations (1) was numerically integrated using the fourth-order Runge–Kutta method. Unstable saddle periodic orbits were separated using the *SD* method as described by Schmelcher *et al.* [15, 16]. In order to study the time shift between the unstable periodic cycles, it is necessary to find synchronized orbits simultaneously in both coupled systems. For this reason, the *SD* method was applied in a six-dimensional phase space formed by the partial three-dimensional phase spaces of the interacting Rössler oscillators.

Figure 1 shows the time series $x_{1,2}(t)$ corresponding to a saddle equiphase resonance m:m cycle with a topological period of $m = 2$ for a coupling parameter of $\varepsilon = 0.07$, whereby coupled systems (1) exhibit phase synchronization. As can be seen, there is a certain time shift between the two curves. Let us consider the given equiphase cycle for various values of the coupling parameter. As ε increases, the time shift Δt between the synchronized orbits decreases; upon the onset of a lag synchronization regime, the time shift, as was noted above, coincides with the time lag between the state vectors of the interacting Rössler oscillators. It is important to note that the dependence of Δt on ε obeys a power law,

$$\Delta t \sim \varepsilon^n, \tag{2}$$

where $n = -1$ in the entire range of variation of the coupling parameter, in which the equiphase resonance saddle 2 : 2 cycle exists. In other words, this relationship is valid in both lag and phase synchronization regimes. An analogous behavior is observed for all equiphase synchronized unstable saddle orbits with other topological periods m. It should emphasized that the time shift $\Delta t(\varepsilon)$ is the same for all equiphase saddle cycles, irrespective of their topological periods m.

Figure 2 shows a double logarithmic plot of the time shift Δt between synchronized unstable equiphase saddle orbits with the topological periods $m = 1$ (*1*), 2 (*2*), and 3 (*3*) versus the coupling parameter ε. From this plot, it is clearly seen that, first, the time shift Δt between the unstable periodic trajectories in the first and second coupled systems as a function of ε is described by the power law (2) with the exponent $n = -1$, irrespective of the topological period. For simplicity, Fig. 2 presents the data only for three equiphase saddle cycles with the minimum topological periods m, but the time shift for the cycles with higher topological periods behaves in the same manner. Second, the time shift Δt for a fixed coupling parameter ε is the same for all synchronized equiphase orbits.

Thus, we have analyzed, in a particular case of two mutually coupled Rössler systems, the time shift between synchronized equiphase unstable saddle orbits incorporated into chaotic attractors of coupled oscillators. It is demonstrated that the dependence of this time shift on the coupling parameter ε is the same for all cycles, irrespective of their topological periods, and is described by the universal power law $\Delta t \sim \varepsilon^{-n}$ with an exponent of $n = -1$.

It should be noted that the obtained results agree well with the conclusions [13] concerning the phase (or time) shift between synchronized spectral components of the Fourier spectra of mutually coupled chaotic oscillators in the course of lag synchronization. At the same time, the problem of interrelation between the behavior of the Fourier components and the dynamics of saddle orbits incorporated into the given chaotic attractor requires further investigation.

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REFERENCES

1. A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Univ. Press, Cambridge, 2001).
2. A. S. Dmitriev and A. I. Panas, *Dynamical Chaos: Novel Information Carriers for Communication Systems* (Fizmatlit, Moscow, 2002) [in Russian].
3. R. C. Elson *et al.*, Phys. Rev. Lett. **81**, 5692 (1998).
4. V. S. Anishchenko, V. V. Astakhov, T. E. Vadivasova, *et al.*, *Nonlinear Effects in Random and Stochastic Systems* (Inst. Komp'yut. Issled., Moscow, 2003) [in Russian].
5. A. A. Koronovskiĭ and A. E. Hramov, Pis'ma Zh. Éksp. Teor. Fiz. **79**, 391 (2004) [JETP Lett. **79**, 316 (2004)].
6. A. E. Hramov and A. A. Koronovskiĭ, Chaos **14**, 603 (2004).
7. A. A. Koronovskiĭ and A. E. Hramov, *Continuous Wavelet Analysis and Its Applications* (Fizmatlit, Moscow, 2003) [in Russian].
8. P. Cvitanović, Physica D **51**, 138 (1991).
9. N. F. Rulkov, Chaos **6**, 262 (1996).
10. A. Pikovsky, G. Osipov, M. Rosenblum, *et al.*, Phys. Rev. Lett. **79**, 47 (1997).
11. A. Pikovsky, M. Zaks, M. Rosenblum, *et al.*, Chaos **7**, 680 (1997).
12. D. Pazó, M. Zaks, and J. Kurths, Chaos **13**, 309 (2002).
13. A. A. Koronovskiĭ, O. I. Moskalenko, and A. E. Hramov, Pis'ma Zh. Éksp. Teor. Fiz. **80**, 25 (2004) [JETP Lett. **80**, 20 (2004)].
14. M. G. Rosenblum *et al.*, Phys. Rev. Lett. **89**, 264102 (2002).
15. P. Schmelcher and F. K. Diakonos, Phys. Rev. E **57**, 2739 (1998).
16. D. Pingel, P. Schmelcher, and F. K. Diakonos, Phys. Rev. E **64**, 026214 (2001).

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