



Time scale synchronization of chaotic oscillators

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Abstract

This paper presents the result of the investigation of chaotic oscillator synchronization. A new approach to detect the synchronized behaviour of chaotic oscillators has been proposed. This approach is based on the analysis of different time scales in the time series generated by the coupled chaotic oscillators. This approach has been applied for the coupled Rössler and Lorenz systems. © 2005 Elsevier B.V. All rights reserved.

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1. Introduction

Synchronization of chaotic oscillators is one of the fundamental phenomena of nonlinear dynamics. It takes place in many physical [1–6] and biological [7–9] processes. It seems to play an important role in the ability of biological oscillators, such as neurons, to act cooperatively [10–12].

There are several different types of synchronization of coupled chaotic oscillators which have been described theoretically and observed experimentally [13–

16]. The *complete synchronization* (CS) implies coincidence of states of coupled oscillators $\mathbf{x}_1(t) \cong \mathbf{x}_2(t)$, the difference between state vectors of coupled systems converges to zero in the limit $t \rightarrow \infty$, [17–20]. It appears when interacting systems are identical. If the parameters of coupled chaotic oscillators slightly mismatch, the state vectors are close $|\mathbf{x}_1(t) - \mathbf{x}_2(t)| \approx 0$, but differ from each other. Another type of synchronized behavior of coupled chaotic oscillators with slightly mismatched parameters is the *lag synchronization* (LS), when shifted in time, the state vectors coincide with each other, $\mathbf{x}_1(t + \tau) = \mathbf{x}_2(t)$. When the coupling between oscillator increases the time lag τ decreases and the synchronization regime tends to be CS one [21–23]. The *generalized synchronization* (GS) [24–26] introduced for drive–response systems, means

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that there is some functional relation between coupled chaotic oscillators, i.e. $\mathbf{x}_2(t) = \mathbf{F}[\mathbf{x}_1(t)]$.

Finally, it is necessary to mention the *phase synchronization* (PS) regime. To describe the phase synchronization the instantaneous phase $\phi(t)$ of a chaotic continuous time series is usually introduced [13–16,27,28]. The phase synchronization means the entrainment of phases of chaotic signals, whereas their amplitudes remain chaotic and uncorrelated.

All synchronization types mentioned above are associated with each other (see for detail [1,24,22]), but the relationship between them is not completely clarified yet. For each type of synchronization there are their own ways to detect the synchronized behavior of coupled chaotic oscillators. The complete synchronization can be displayed by means of comparison of system state vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, whereas the lag synchronization can be determined by means of the similarity function [21]

$$S^2(\tau) = \frac{\langle |\mathbf{x}_2(t + \tau) - \mathbf{x}_1(t)|^2 \rangle}{\sqrt{\langle |\mathbf{x}_1(t)|^2 \rangle \langle |\mathbf{x}_2(t)|^2 \rangle}}. \tag{1}$$

If the lag synchronization regime takes place the similarity function $S(\tau)$ has its minimum $\sigma = \min_{\tau} S(\tau) = 0$ for τ corresponding to the time shift between the state vectors.¹

The case of the generalized synchronization is more intricate because the functional relation $\mathbf{F}[\cdot]$ can be very complicated, but there are several methods to detect the synchronized behavior of coupled chaotic oscillators, such as the auxiliary system approach [29] or the method of nearest neighbors [24,30].

Finally, the phase synchronization of two coupled chaotic oscillators occurs if the difference between the instantaneous phases $\phi(t)$ of chaotic signals $\mathbf{x}_{1,2}(t)$ is bounded by some constant

$$|\phi_1(t) - \phi_2(t)| < \text{const}. \tag{2}$$

It is possible to define a mean frequency

$$\bar{\Omega} = \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = \langle \dot{\phi}(t) \rangle, \tag{3}$$

which should be the same for both coupled chaotic system, i.e., the phase locking leads to the frequency

¹ It is clear, that for the case of the complete synchronization $S(\tau)$ reaches minimum value $\sigma = 0$ for $\tau = 0$.

entrainment. It is important to notice, to obtain correct results the mean frequency $\bar{\Omega}$ of chaotic signal $\mathbf{x}(t)$ should coincide with the main frequency $\Omega_0 = 2\pi f_0$ of the Fourier spectrum (for detail, see [31]). Unfortunately, there is no general way to introduce the phase for chaotic time series. There are several approaches which allow to define the phase for “good” systems with simple topology of chaotic attractor (so-called “phase coherent attractor”), the Fourier spectrum of which contains the single main frequency f_0 . The example of attractor and Fourier spectrum of such “good” system is shown in Fig. 1.

First of all, the instantaneous phase $\phi(t)$ can be introduced as an angle in polar coordinates on the (x, y) -plane [32,21]

$$\phi = \arctan \frac{y}{x}, \tag{4}$$

but for those all trajectories of chaotic attractor projection on the (x, y) -plane should revolve around some origin. Sometimes, a coordinate transformation can be used to obtain a proper projection [32,13]. Another way to define the phase $\phi(t)$ of chaotic time series $x(t)$ is the constructing of the analytical signal [15,27]

$$\zeta(t) = x(t) + j\tilde{x}(t) = A(t)e^{j\phi(t)}, \tag{5}$$

where the function $\tilde{x}(t)$ is the Hilbert transform of $x(t)$

$$\tilde{x}(t) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{+\infty} \frac{x(\tau)}{t - \tau} d\tau \tag{6}$$

(where P.V. means that the integral is taken in the sense of the Cauchy principal value). The instantaneous phase $\phi(t)$ is defined from (5) and (6). Moreover, the Poincaré secant surface can be used for the introducing of the instantaneous phase of chaotic dynamical system [15,27]

$$\phi(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, t_n \leq t \leq t_{n+1}, \tag{7}$$

where t_n is the time of the n th crossing of the secant surface by the trajectory. Finally, the phase of chaotic time series can be introduced by means of the continuous wavelet transform [33], but the appropriate wavelet function and its parameters should be chosen [34].

All these approaches give correct results for “good” systems with well-defined phase, but fail for oscillators with non-revolving trajectories. Such chaotic oscillators are often called as “systems with ill-defined

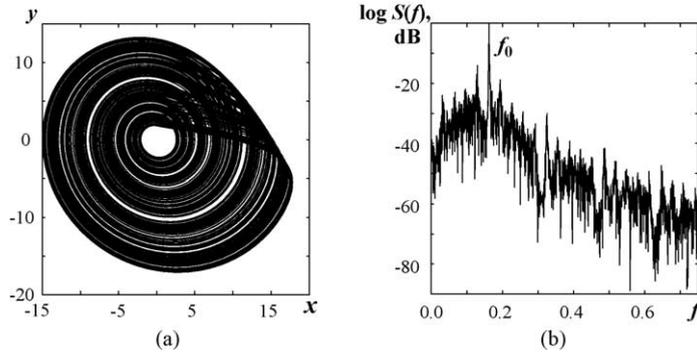


Fig. 1. (a) Phase coherent attractor and (b) spectrum of the first Rössler system (15). Coupling parameter ε between oscillators is equal to zero.

phase”. The phase introducing based on the approaches mentioned above for the system with ill-defined phase leads usually to incorrect results [31]. Therefore, the phase synchronization of such systems can be usually detected by means of the indirect indications [32,35] and measurements [36].

In this paper we propose a new approach to detect the synchronization between two coupled chaotic oscillators. The main idea of this approach consists in the analysis of the system behavior on different time scales that allows us to consider different cases of synchronization from the universal positions. Using the continuous wavelet transform [37–40] we introduce the time scales s and associated with them instantaneous phases $\phi_s(t)$. As we will show further, if two chaotic oscillators demonstrate any type of synchronized behavior, in the time series $\mathbf{x}_{1,2}(t)$ generated by these systems there are time scales s necessarily correlated for which the phase locking condition

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \text{const} \quad (8)$$

is satisfied.

The structure of this paper is the following. In Section 2 we discuss the continuous wavelet transform and the method of the time scales s and associated with them phases $\phi_s(t)$ definition. In Section 3 we consider the case of the phase synchronization of two mutually coupled Rössler systems. We demonstrate the application of our method and discuss the relationship between our and traditional approaches. Section 4 deals with the synchronization of two mutually coupled Rössler systems with funnel attractors. In this case the traditional methods for phase introducing fail and there is no pos-

sibility to detect the phase synchronization regime, respectively. The synchronization between systems can be revealed here only by means of the indirect measurements (see for detail [36]). We demonstrate the efficiency of our method for such cases. In Section 5 we consider application of our method for the unidirectional coupled Rössler and Lorenz systems when the generalized synchronization is observed. In Section 6 we discuss the correlation between different types of the chaotic synchronization. The final conclusion is presented in Section 7.

2. Continuous wavelet transform

Let us consider continuous wavelet transform of chaotic time series $x(t)$

$$W(s, t_0) = \int_{-\infty}^{+\infty} x(t) \psi_{s,t_0}^*(t) dt, \quad (9)$$

where $\psi_{s,t_0}(t)$ is the wavelet-function related to the mother-wavelet $\psi_0(t)$ as

$$\psi_{s,t_0}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-t_0}{s}\right). \quad (10)$$

The time scale s corresponds to the width of the wavelet function $\psi_{s,t_0}(t)$, and t_0 is shift of wavelet along the time axis, the symbol “*” in (9) denotes complex conjugation. It should be noted that the time scale s is usually used instead of the frequency f of Fourier transformation and can be considered as the quantity inverted to it.

The Morlet-wavelet [41]

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}} \exp(j\Omega_0\eta) \exp\left(\frac{-\eta^2}{2}\right) \quad (11)$$

has been used as a mother-wavelet function. The choice of parameter value $\Omega_0 = 2\pi$ provides the relation $s = 1/f$ between the time scale s of wavelet transform and frequency f of Fourier transformation.

The wavelet surface

$$W(s, t_0) = |W(s, t_0)|e^{j\phi_s(t_0)} \quad (12)$$

describes the system's dynamics on every time scale s at the moment of time t_0 . The value of $|W(s, t_0)|$ indicates the presence and intensity of the time scale s mode in the time series $x(t)$ at the moment of time t_0 . It is possible to consider the quantities

$$E(s, t_0) = |W(s, t_0)|^2 \quad (13)$$

and

$$\langle E(s) \rangle = \int |W(s, t_0)|^2 dt_0, \quad (14)$$

which are instantaneous and integral energy distributions on time scales, respectively.

At the same time, the phase $\phi_s(t) = \arg W(s, t)$ is naturally introduced for every time scale s . It means that it is possible to describe the behavior of each time scale s by means of its own phase $\phi_s(t)$. If two interacting chaotic oscillators are synchronized it means that in time series $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ there are scales s correlated with each other. To detect such correlation one can examine the condition (8) which should be satisfied for synchronized time scales.

3. Phase synchronization of two Rössler systems

Let us start our consideration with two mutually coupled Rössler systems with slightly mismatched parameters [27,28]

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2}, \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \quad (15)$$

where $a = 0.165$, $p = 0.2$, and $c = 10$. The parameters $\omega_{1,2} = \omega_0 \pm \Delta$ determine the parameter mistuning, ε is the coupling parameter. In [21] it has been shown that

for these control parameter values and coupling parameter $\varepsilon = 0.05$ the phase synchronization is observed.

For this case the phase of chaotic signal can be easily introduced in one of the ways (4)–(7) mentioned above, because the phase coherent attractor with rather simple topological properties is realized in the system phase space. The attractor projection on the (x, y) -plane resembles the smeared limit cycle where the phase point always rotates around the origin (Fig. 1a). The Fourier spectrum $S(f)$ contains the main frequency peak $f_0 \simeq 0.163$ (see Fig. 1b) which coincides with the mean frequency $\bar{f} = \Omega/2\pi$, determined from the instantaneous phase $\phi(t)$ dynamics (3). Therefore, there are no complications to detect the phase synchronization regime in two coupled Rössler systems (15) by means of traditional approaches.

When the coupling parameter ε is equal to 0.05 the phase synchronization between chaotic oscillators is observed. The phase locking condition (2) is satisfied and the mean frequencies $\Omega_{1,2}$ are entrained. So, the time scales $s_0 \simeq 6$ of both chaotic systems corresponding to the mean frequencies $\Omega_{1,2}$ should be correlated with each other. Correspondingly, the phases $\phi_{s_{1,2}}(t)$ associated with these time scales s should be locked and the condition (8) should be satisfied. The time scales which are the nearest to the time scale s_0 should also be correlated, but the interval of the correlated time scales depends upon the coupling strength. At the same time, there should be time scales which remain uncorrelated. These uncorrelated time scales cause the difference between chaotic oscillations of coupled systems.

Fig. 2 illustrates the behavior of different time scales for two coupled Rössler systems (15) with phase coherent attractors. It is clear, that the phase difference $\phi_{s_1}(t) - \phi_{s_2}(t)$ for scales $s_0 = 6$ is bounded and therefore time scales $s_0 = 6$ corresponding to the main frequency of Fourier spectrum f_0 are synchronized. It is important to note that wavelet power spectra $\langle E_{1,2}(s) \rangle$ close to each other (see Fig. 2a) and time scales s characterized by the large value of energy (e.g., $s = 5$) close to the main time scale $s_0 = 6.0$ are correlated, too. There are also time scales which are not synchronized, like $s = 3.0$, $s = 4.0$, etc. (see Fig. 2b).

So, when two mutually coupled chaotic oscillators with phase coherent attractors are considered, both the traditional method based on the instantaneous phase $\phi(t)$ of chaotic signal introducing and our approach lead to the equivalent results.

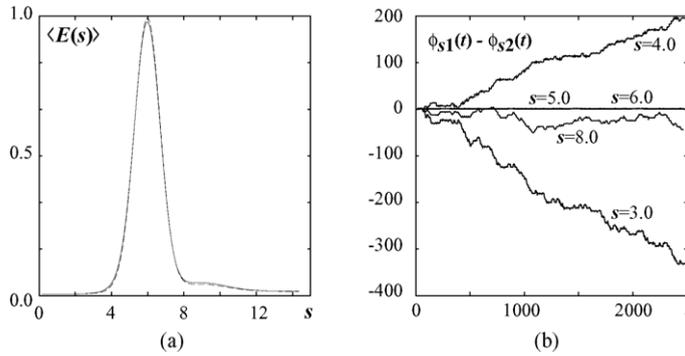


Fig. 2. (a) Wavelet power spectrum $\langle E(s) \rangle$ for the first (solid line) and the second (dashed line) Rössler systems (15). (b) The dependence of phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ on time t for different time scales s . The coupling parameter between oscillators is $\varepsilon = 0.05$. The phase synchronization for two coupled chaotic oscillators is observed.

4. Synchronization of two Rössler systems with funnel attractors

Let us consider the more complicated example when it is impossible to correctly introduce the instantaneous phase $\phi(t)$ of chaotic signal $\mathbf{x}(t)$. It is clear, that for such cases the traditional methods of the phase synchronization detecting fail and it is necessary to use the other techniques, e.g., indirect measurements [36]. On the contrary, our approach gives correct results and allows to detect the synchronization between chaotic oscillators as easily as before.

To illustrate it we consider two non-identical coupled Rössler systems with funnel attractors (Fig. 3):

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \quad (16)$$

where ε is a coupling parameter, $\omega_1 = 0.98, \omega_2 = 1.03$. The control parameter values have been selected by analogy with [36] as $a = 0.22, p = 0.1, c = 8.5$.

In [36] it has been shown by means of the indirect measurements that for the coupling parameter value $\varepsilon = 0.05$ the synchronization of two mutually coupled Rössler systems (16) takes place. Our approach based on the analysis of the dynamics of different time scales s gives analogous results. So, the behavior of the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for this case has been presented in Fig. 4b. One can see that the phase locking takes place for the time scales $s = 5.25$ which are characterized by the largest energy value in the wavelet power spectra $\langle E(s) \rangle$ (see Fig. 4a). Thus, we can say that the time scales $s = 5.25$ of two oscillators are synchronized with each other. It is important to note that the other time scales (e.g., $s = 4.5, 6.0$ et al.) remain uncorrelated. For such time

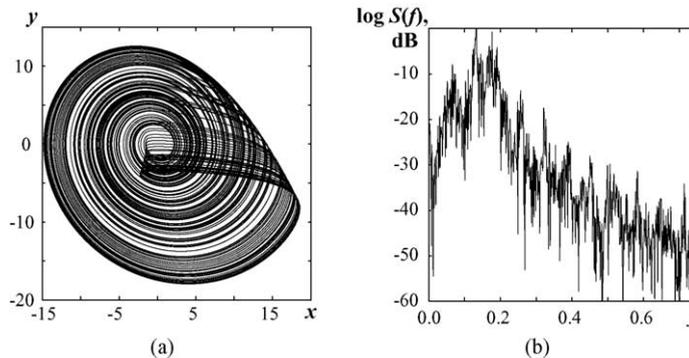


Fig. 3. (a) Phase picture and (b) power spectrum of the first Rössler system (16) oscillations. Coupling parameter ε is equal to zero.

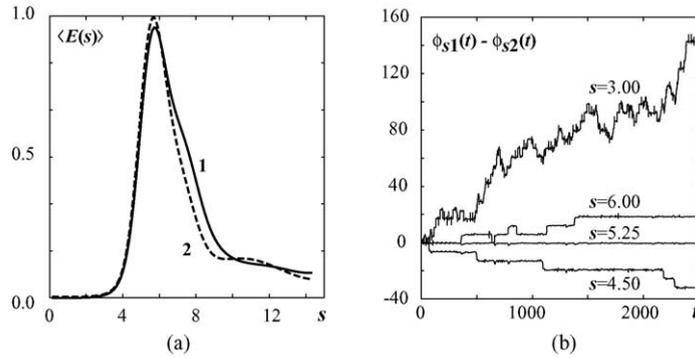


Fig. 4. (a) The normalized energy distribution in wavelet spectrum $\langle E(s) \rangle$ for the first (the solid line denoted “1”) and the second (the dashed line denoted “2”) Rössler systems (16); (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for two coupled Rössler systems. The value of coupling parameter has been selected as $\varepsilon = 0.05$. The time scales $s = 5.25$ are correlated with each other and the synchronization has been observed.

scales the phase locking has not been observed (see Fig. 4b).

It is clear, that the mechanism of the synchronization of coupled chaotic oscillators is the same in both cases considered in Sections 3 and 4. The synchronization phenomenon is caused by the existence of time scales s in system dynamics correlated with each other. Therefore, there is no reason to divide the considered synchronization examples into different types.

5. Generalized synchronization of Rössler and Lorenz systems

Let us consider another type of synchronized behavior, so-called the generalized synchronization. As oscillator samples the Rössler (drive) and Lorenz (response) systems have been selected. The equations of the drive system $\mathbf{x}_1 = (x_1, y_1, z_1)^T$ are

$$\begin{aligned} \dot{x}_1 &= -\omega y_1 - z_1, \\ \dot{y}_1 &= \omega x_1 + a y_1, \\ \dot{z}_1 &= p + z_1(x_1 - c), \end{aligned} \quad (17)$$

and the response system $\mathbf{x}_2 = (x_2, y_2, z_2)^T$ is given by

$$\begin{aligned} \dot{x}_2 &= -\sigma(x_2 - y_2), \\ \dot{y}_2 &= r u(t) - y_2 - u(t) z_2, \\ \dot{z}_2 &= u(t) y_2 - b z_2. \end{aligned} \quad (18)$$

The drive signal $u(t)$ and control parameter values have been selected by analogy with [25] as $u(t) = x_1 +$

$y_1 + z_1$, $p = 2$, $c = 4$, $\omega = 1$, $a = 1$, $\sigma = 10$, $r = 28$, $b = 2.666$. It has been analytically shown (see [25]) that the response system (18) is asymptotically stable for arbitrary drive signals $u(t)$ and arbitrary initial conditions, and therefore the generalized synchronization always occurs although drive and response systems are completely different. Obviously, the generalized synchronization regime can also be detected by means of other numerical or experimental methods (e.g., the auxiliary system approach).

The chaotic attractors and corresponding them power spectra $S(f)$ of unidirectional coupled Rössler and Lorenz systems are shown in Figs. 5 and 6, respectively. One can see the presence of the peaks corresponding to the frequencies of the Rössler system oscillations in the Fourier spectrum of the Lorenz system (see Figs. 5b and 6b). Therefore, the time scales s of coupled systems can be correlated with each other if chaotic oscillators are synchronized on these time scales.

The dependencies of phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ on time for different time scales s are shown in Fig. 7. It is clear, that there is the range of scales (approximately $s = 5 \div 7$) whose the behavior is synchronized. At the same time there are time scales which are not synchronized, as in the case of the phase synchronization (see. e.g., Fig. 2). It is important to note, despite the fact that there is the frequency $f_R \approx 0.33$ in the spectra of both Rössler and Lorenz systems (see Figs. 5b and 6b), the time scales $s_R = 1/f_R \approx 3$ of coupled oscillators are not synchronized (Fig. 7b). So, the presence of equivalent frequencies f in the power

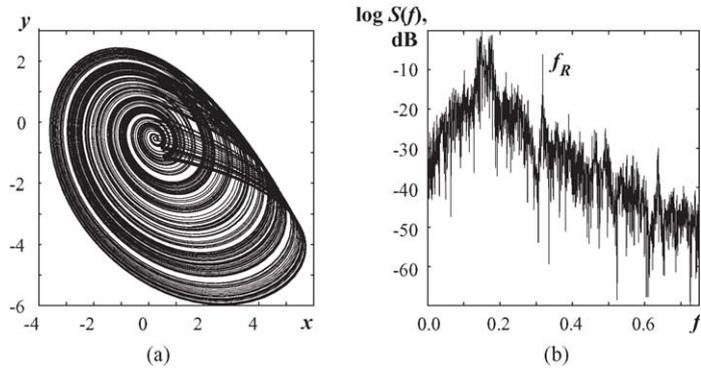


Fig. 5. (a) Chaotic attractor and (b) spectrum of the Rössler (drive) system (17).

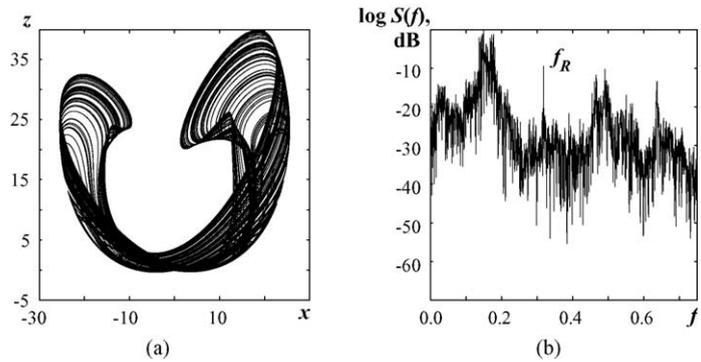


Fig. 6. (a) Chaotic attractor and (b) spectrum of the Lorenz (response) system (18).

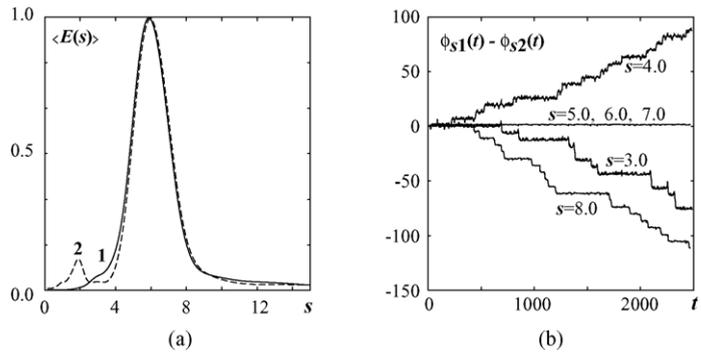


Fig. 7. (a) The normalized energy distribution in wavelet spectrum $\langle E(s) \rangle$ for the Rössler (17) (the solid line denoted “1”) and Lorenz (18) (the dashed line denoted “2”) systems. Each energy distribution has been normalized on its own coefficient; (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for coupled systems, when the generalized synchronization regime takes place. The time scales $s = 5$, $s = 6$, $s = 7$ are correlated with each other whereas the other time scales (e.g., $s = 3$, $s = 8$, etc.) are not synchronized.

spectra of interacting systems does not mean the obligatory synchronization on the time scale $s_f = 1/f$ corresponding to this frequency.

Thus, the generalized synchronization of two unidirectionally coupled absolutely different systems is revealed as the synchronous dynamics of some time

scales on which the phases $\phi_s(t)$ are locked. One can see that different types of the synchronization such as PS and GS are quite similar to each other when the time scale behavior of interacting systems is analysed. So, the time scales of Rössler and Lorenz systems in the generalized synchronization regime behave similarly to the cases of the phase synchronization of two Rössler systems (15), although for the considered generalized synchronization case we can not introduce correctly the instantaneous phase $\phi(t)$ for either Rössler or Lorenz systems (see Fig. 5 and 6) at all. Obviously, it is necessary to consider the correlation between different types of synchronization and transitions from one to another. This topic will be discussed in the next section.

6. From unsynchronized behavior to complete synchronization regime

It has been shown in [21] that there is certain relationship between PS, LS and CS for chaotic oscillators with slightly mismatched parameters. With the increase of coupling strength the systems undergo the transition from unsynchronized chaotic oscillations to the phase synchronization. With a further increase of coupling the lag synchronization is observed. Further increasing of the coupling parameter leads to the decreasing of the time lag and both systems tend to have the complete synchronization regime.

Let us consider the dynamics of different time scales s of two nonidentical mutually coupled Rössler systems (16). If there is no phase synchronization between

the oscillators, then their dynamics remain uncorrelated for all time scales s . Fig. 8 illustrates the dynamics of two coupled Rössler systems when the coupling parameter ε is small enough ($\varepsilon = 0.025$). The power spectra $\langle E(s) \rangle$ of wavelet transform for Rössler systems differ from each other (Fig. 8a), but the maximum values of the energy correspond approximately to the same time scale s in both systems. It is clear, that the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ is not bounded for all time scales (see Fig. 8b). It means that there are no time scales s correlated with each other and two Rössler systems are not synchronized at all.

As soon as any of the time scales of the first chaotic oscillator becomes correlated with the other one of the second oscillator (e.g., when the coupling parameter increases), the phase synchronization occurs (see Fig. 4 in Section 4). It is clear, that the time scales s characterized by the largest value of energy in wavelet spectrum $\langle E(s) \rangle$ become correlated first. The other time scales remain uncorrelated as before. The phase synchronization between chaotic oscillators leads to the phase locking (8) on the correlated time scales s .

When the parameter of coupling between chaotic oscillators increases, more and more time scales become correlated and one can say that the degree of the synchronization grows. So, with the further increasing of the coupling parameter value (e.g., $\varepsilon = 0.07$) in the coupled Rössler systems (16) the time scales which were uncorrelated before become synchronized (see Fig. 9b). It is evident, that the time scales $s = 4.5$ are synchronized in comparison with the previous case ($\varepsilon = 0.05$, Fig. 4b) when these time scales were uncor-

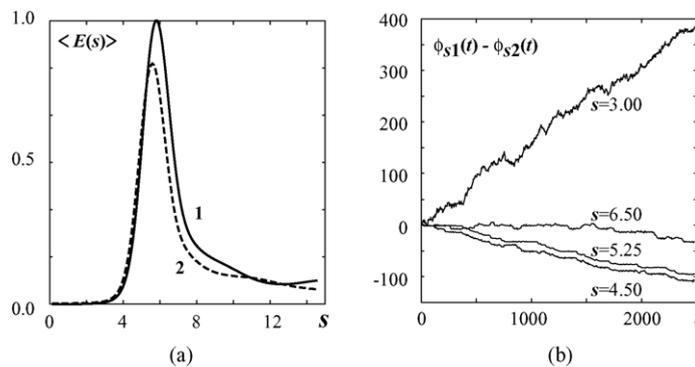


Fig. 8. (a) The normalized energy distribution in wavelet spectrum $\langle E(s) \rangle$ for the first (the solid line denoted “1”) and the second (the dashed line denoted “2”) Rössler systems; (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for two coupled Rössler systems. The value of coupling parameter has been selected as $\varepsilon = 0.025$. There is no phase synchronization between systems.

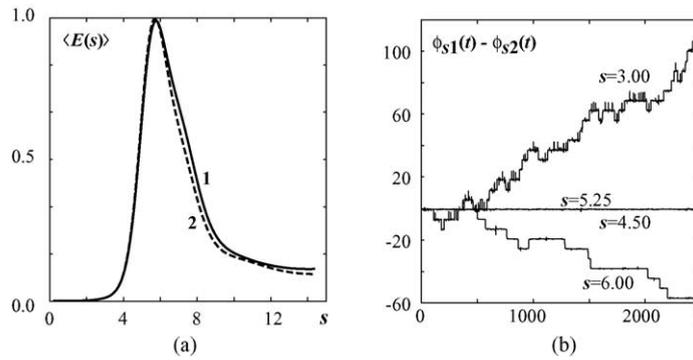


Fig. 9. (a) The normalized energy distribution in wavelet spectrum ($E(s)$) for the first (the solid line denoted “1”) and the second (the dashed line denoted “2”) Rössler systems; (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for two coupled Rössler systems. The value of coupling parameter has been selected as $\varepsilon = 0.07$.

related. The number of time scales s demonstrating the phase locking increases, but there are nonsynchronized time scales as before (e.g., the time scales $s = 3.0$ and $s = 6.0$ remain still nonsynchronized).

Arising of the lag synchronization [21] between oscillators means that all time scales are correlated. Indeed, from the condition of the lag-synchronization $x_1(t - \tau) \simeq x_2(t)$ one can obtain that $W_1(s, t - \tau) \simeq W_2(t, s)$ and therefore $\phi_{s1}(t - \tau) \simeq \phi_{s2}(t)$. It is clear, in this case the phase locking condition (8) is satisfied for all time scales s . For instance, when the coupling parameter of chaotic oscillators (16) becomes large enough ($\varepsilon = 0.25$) the lag synchronization of two coupled oscillators appears. In this case the power spectra of wavelet transform coincide with each other (see Fig. 10a)

and the phase locking takes place for all time scales s (Fig. 10b). It is important to note that the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ is not equal to zero for the case of the lag synchronization. It is clear that this difference depends on the time lag τ .

Further increasing of the coupling parameter leads to the decreasing of the time lag τ [21]. Both systems tend to have the complete synchronization regime $x_1(t) \simeq x_2(t)$, so the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ tends to be a zero for all time scales.

It is clear, that PS, LS and CS are naturally interrelated with each other and the synchronization type depends on the number of synchronized time scales. The details of the relations between PS and GS is not at all clear. There are several works [1,22] dealing with the problem, how GS and PS are correlated with each other.

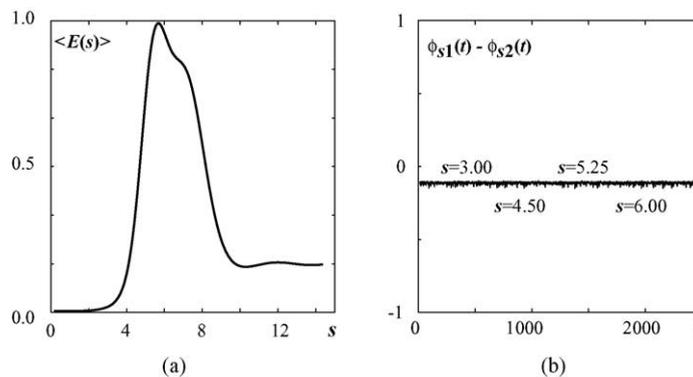


Fig. 10. (a) The normalized energy distribution in wavelet spectrum ($E(s)$) for the Rössler system; (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for two coupled Rössler systems. The value of coupling parameter has been selected as $\varepsilon = 0.25$. The lag synchronization has been observed, all time scales are synchronized.

For instance, in [22] it has been reported that two unidirectional coupled Rössler systems can demonstrate the generalized synchronization while the phase synchronization has not been observed. This case allows to be explained easily by means of the time scale analysis. The equations of Rössler system are

$$\begin{aligned}\dot{x}_1 &= -\omega_1 y_1 - z_1, \\ \dot{y}_1 &= \omega_1 x_1 + a y_1, \\ \dot{z}_1 &= p + z_1(x_1 - c) \\ \dot{x}_2 &= -\omega_2 y_2 - z_2 + \varepsilon(x_1 - x_2), \\ \dot{y}_2 &= \omega_2 x_2 + a y_2, \\ \dot{z}_2 &= p + z_2(x_2 - c),\end{aligned}\quad (19)$$

where $\mathbf{x}_1 = (x_1, y_1, z_1)^T$ and $\mathbf{x}_2 = (x_2, y_2, z_2)^T$ are the state vectors of the first (drive) and the second (response) Rössler systems, respectively. The control parameter values have been chosen as $\omega_1 = 0.8$, $\omega_2 = 1.0$, $a = 0.15$, $p = 0.2$, $c = 10$ and $\varepsilon = 0.2$. The generalized synchronization takes place in this case (see [22] for detail). Why it is impossible to detect the phase synchronization in the system (19) despite the generalized synchronization is observed becomes clear from the time scale analysis.

Let us consider Fourier spectra of coupled chaotic oscillators (see Fig. 11). There are two main spectral components with frequencies $f_1 = 0.125$ and $f_2 = 0.154$ in these spectra. The analysis of behavior of time scales has shown that both the time scales $s_1 = 1/f_1 = 8$ of coupled oscillators corresponding to the frequency f_1 and time scales close to s_1 are synchronized while

the time scales $s_2 = 1/f_2 \simeq 6.5$ and close to them do not demonstrate synchronous behavior (Fig. 12b).

The source of such behavior of time scales becomes clear from the wavelet power spectra $\langle E(s) \rangle$ of both systems (see Fig. 12a). The time scale s_1 of the drive Rössler system is characterized by the large value of energy while the part of energy being fallen on this scale of the response system is quite small. Therefore, the drive system dictates its own dynamics on the time scale s_1 to the response system. The opposite situation takes place for the time scales s_2 (see Fig. 12a). The drive system can not dictate its dynamics to the response system because the part of energy being fallen on this time scale is small in the first Rössler system and large enough in the second one. So, time scales s_2 are not synchronized.

Thus, the generalized synchronization of the unidirectional coupled Rössler systems appears as the time scale synchronized dynamics, as another synchronization types does before. It is also clear, why the phase synchronization has not been observed in this case. Fig. 11 shows that the instantaneous phases $\phi_{1,2}(t)$ of chaotic signals $\mathbf{x}_{1,2}(t)$ introduced by means of traditional approaches (4)–(7) are determined by both frequencies f_1 and f_2 , but only the spectral components with the frequency f_1 are synchronized. So, the observation of instantaneous phases $\phi_{1,2}(t)$ does not allow to detect the phase synchronization in this case although the synchronization of time scales takes place.

Thus, one can see that there is a close relationship between different types of the chaotic oscillator syn-

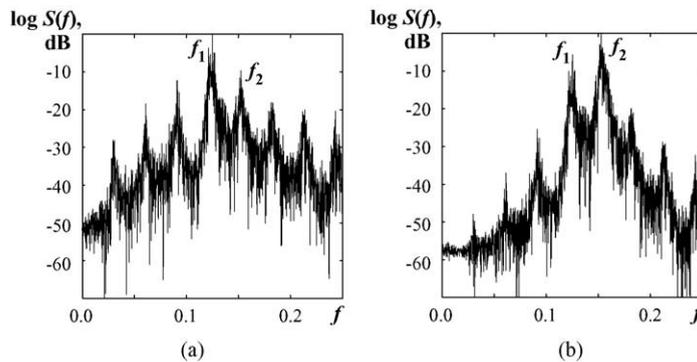


Fig. 11. Fourier spectra for (a) the first (drive) and (b) the second (response) Rössler systems (19). The coupling parameter is $\varepsilon = 0.2$. The generalized synchronization takes place.

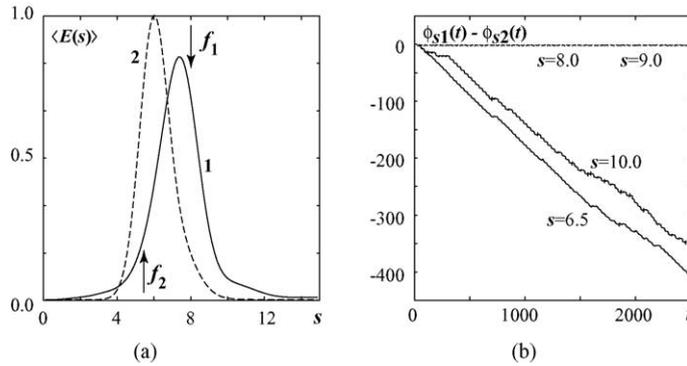


Fig. 12. (a) The normalized energy distribution in wavelet spectrum $\langle E(s) \rangle$ for the first (the solid line denoted “1”) and the second (the dashed line denoted “2”) Rössler systems. The time scales pointed with arrows correspond to the frequencies $f_1 = 0.125$ and $f_2 = 0.154$, respectively; (b) the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for two coupled Rössler systems. The generalized synchronization has been observed.

chronization. Unfortunately, it is not clear, how one can distinguish the phase synchronization² and the generalized synchronization using only the results obtained from the analysis of the time scale dynamics. Neither can be said anything about the character of the relationship between these synchronization types is. We suppose that GS and PS are often practically equivalent (when, of course, it is possible to define correctly the instantaneous phase of chaotic signal by means of traditional technique). Nevertheless, this problem should be investigated separately later.

7. Conclusions

Summarizing this work we would like to note several principal aspects. Firstly, the traditional approach for the detecting the phase synchronization based on the introducing of the instantaneous phase $\phi(t)$ of chaotic signal is suitable and correct for such time series characterized by the Fourier spectrum with the single main frequency f_0 . In this case the phase ϕ_{s0} associated with the time scale s_0 corresponding to the main frequency f_0 of the Fourier spectrum coincides approximately with the instantaneous phase $\phi(t)$ of chaotic signal introduced by means of the traditional approaches (see

also [34]). Indeed, as the other frequencies (the other time scales) do not play a significant role in the Fourier spectrum, the phase $\phi(t)$ of chaotic signal is close to the phase $\phi_{s0}(t)$ of the main spectral frequency f_0 (and the main time scale s_0 , respectively). It is obvious, that in this case the mean frequencies $\bar{f} = \langle \dot{\phi}(t) \rangle / 2\pi$ and $\bar{f}_{s0} = \langle \dot{\phi}_{s0}(t) \rangle / 2\pi$ should coincide with each other and with the main frequency f_0 of the Fourier spectrum (see also [31])

$$\bar{f} = \bar{f}_{s0} = f_0. \quad (20)$$

If the chaotic time series is characterized by the Fourier spectrum without the main single frequency (like the spectrum shown in the Fig. 3b) the traditional approaches (4)–(7) fail. One has to consider the dynamics of the system on all time scales, but it is impossible to do it by means of the instantaneous phase $\phi(t)$ introduced as (4)–(7). On the contrary, our approach based on the continuous wavelet transform can be used for both types of chaotic signals.

Secondly, our approach can be easily applied to the experimental data because it does not require any a-priori information about the considered dynamical systems. Moreover, in several cases the influence of the noise can be reduced by means of the wavelet transform (for detail, see [37,42,43]). We believe that our approach will be useful and effective for the analysis of physical, biological, physiological and other data, such as [10,34,33].

² Here we mean the phase synchronization between chaotic oscillators takes place if the instantaneous phase $\phi(t)$ of chaotic signal may be correctly introduced by means of (4)–(7) and the phase locking condition (2) is satisfied.

Finally, it is important to note that analysis of the system dynamics on the different time scales based on the continuous wavelet transform allows to consider the different types of behavior of coupled oscillators (such as the complete synchronization, the lag synchronization, the phase synchronization, the generalized synchronization and the nonsynchronized oscillations) from the universal position. It is clear, that the number of synchronized time scales determines the type of behavior uniquely. Probably, the quantitative characteristic of the synchronization measure can be introduced. This method (with insignificant modifications) can also be applied to dynamical systems synchronized by the external (e.g., harmonic) signal.

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