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Relationship between Phase Synchronization of Chaotic Oscillators and Time Scale Synchronization

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Abstract—The relationship between the phase synchronization of chaotic oscillators and their time scale synchronization has been considered, and it is established that the onset of the time scale synchronization depends on the resolving power of the base wavelet. This synchronization usually appears either before or (if said resolving power is insufficient) simultaneously with the onset of phase synchronization. The results are illustrated in the case of one-way coupled Rössler type dynamical systems. © *2005 Pleiades Publishing, Inc.*

Chaotic synchronization of dynamical systems is among basic nonlinear phenomena which have been extensively studied in recent years [1] and are also of considerable practical importance. The development of the theory of dynamical chaos revealed various types of synchronous chaotic behavior of coupled dynamical systems, including the phase [1], generalized [2], lag [3], and complete synchronization [4]. Each particular type of synchronous chaotic dynamics has its own distinctive features, but the question of the relationships between various types of synchronous behavior has been also actively discussed in recent years. Indeed, various forms or synchronization of coupled chaotic oscillators can be considered as manifestations of certain general laws valid for coupled nonlinear systems (see, e.g., [5–7]).

Recently, we have demonstrated [8–10] that all the aforementioned types of synchronous chaotic behavior can be considered as partial cases of the same type of synchronous dynamics referred to as the time scale synchronization. For example, it was found that, if two coupled chaotic oscillators occur in the regime of phase synchronization, then the time scale synchronization can be revealed as well. The inverse statement is generally not valid and it is quite a typical situation when the coupled chaotic oscillators exhibit signs of the time scale synchronization, while the phase synchronization is not observed (see [8, 9, 11]).

The aim of this study was to establish a relationship between the phase synchronization of chaotic oscillators and their time scale synchronization. In particular, we studied the mutual arrangement of the boundaries of these regimes on the plane of control parameters (drive frequency–coupling parameter) of two coupled oscillators.

The phenomenon of phase synchronization is usually described and analyzed in terms of the phase $\phi(t)$ of the chaotic signal [1, 12–16]. The phase synchronization of chaotic signals implies entrainment of their phases, while the amplitudes remain mutually independent and appear as chaotic. The entrainment of phases leads to coincidence of the frequencies of chaotic signals (the characteristic frequency of a chaotic signal is defined as the average rate $\langle \dot{\phi}(t) \rangle$ of phase variation).

There are several methods of introducing the phase of chaotic oscillations, which are applicable to systems with a simple topology of chaotic attractors (i.e., with phase-coherent attractors). First, the phase $\phi(t)$ of a chaotic signal can be defined as an angle in the polar coordinate system on the (x, y) plane [3, 17]. However, in this case, all trajectories of the projection of the chaotic attractor onto the (x, y) plane will rotate around the origin. Sometimes it is possible to use a transformation of coordinates in order to obtain a projection suited for the phase introduction [1, 17]; otherwise, it is possible to consider the system dynamics on the plane of velocities $(\dot{x}, \dot{y})$ [18]. Another way of defining the phase on a chaotic dynamical system is based on the introduction of an analytical signal [12, 14] with the aid of the Hilbert transform. Finally, the phase of a chaotic signal can be introduced with the aid of the Poincaré cross section [12, 14]. All these approaches give similar correct results for the systems with phase-coherent attractors [17]. On the other hand, these methods frequently lead to incorrect results when applied to the systems with poorly defined phase (see, e.g., [1, 19]). For this reason, the phase synchronization of such a system can be revealed in some cases by means of indirect observations [17, 20] and measurements [21], although there are cases where the presence or absence of phase synchronization cannot be practically established.

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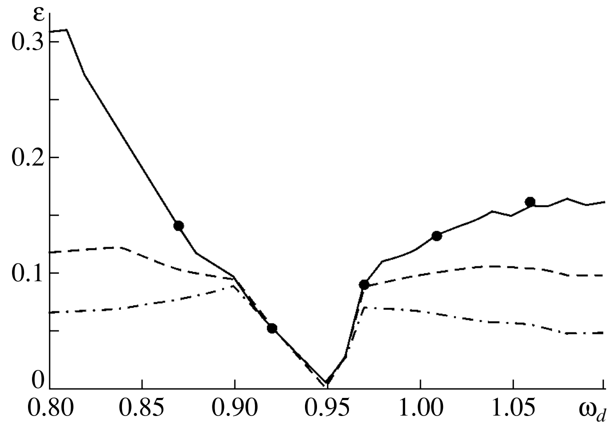

The plane of control parameters (ω_d, ε) for two one-way coupled chaotic Rössler oscillators (3). Solid curve shows the boundary of the onset of phase synchronization of chaotic oscillators; dashed curve shows the boundary of the time scale synchronization for the Morlet wavelet parameter $\Omega = 2\pi$, and dash-dot curve shows this boundary for $\Omega = 16$; points indicate the boundary of the time scale synchronization as determined by analysis using the Morlet wavelet parameter $\Omega = 0.05$.

The regime of time scale synchronization [8–11] is usually described in terms of the time scales s introduced using a continuous wavelet transform with a base wavelet function of the Morlet type (see, e.g., [22]):

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}}\exp(j\Omega\eta)\exp\left(-\frac{\eta^2}{2}\right). \qquad (1)$$

Each time scale s is characterized by the corresponding instantaneous phase $\phi_s(t) = \arg W(s, t)$, where $W(s, t)$ is the complex wavelet surface. In these terms, the phenomenon of time scale synchronization of coupled chaotic systems is manifested by the synchronous behavior of phases $\phi_{s1,2}(t)$ of the first and second oscillator, respectively, observed in the interval of synchronized time scales $s_m < s < s_b$. Each time scale s from this interval obeys the condition of phase entrainment

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \text{const}, \qquad (2)$$

and the energy fraction of the wavelet spectrum in said interval is nonzero. Since the base wavelet (1) depends on the parameter Ω, it is important to study the influence of this parameter on the possibility to determine the boundary of the regime of time scale synchronization.

Let us consider the problem of the relationship between the phase synchronization and the time scale synchronization in the case of two one-way coupled chaotic oscillators representing Rössler systems with slightly mismatched parameters:

$$\dot{x}_d = -\omega_d y_d - z_d, \quad \dot{x}_r = -\omega_r y_r - z_r + \varepsilon(x_d - x_r),$$
$$\dot{y}_d = \omega_d x_d + a y_d, \quad \dot{y}_r = \omega_r x_r + a y_r, \qquad (3)$$
$$\dot{z}_d = p + z_d(x_d - c), \quad \dot{z}_r = p + z_r(x_r - c),$$

where ε is the coupling parameter. The values of the control parameters are selected by analogy with those used in [23]: $a = 0.15$, $p = 0.2$, and $c = 10.0$. The control parameter of the driven (response) system, $\omega_r = 0.95$, which characterizes its fundamental frequency, is fixed. The analogous parameter of the drive system ω_d is varied in the interval from 0.8 to 1.1, which corresponds to a slight mismatch of the two oscillators. It is known that, for the selected values of control parameters, the chaotic attractors of interacting oscillators are phase-coherent (except for small regions on the (ω_d, ε) plane not having significant influence on the final results), which makes possible correct introduction of the phase of chaotic signals and reliable detection of the regime of phase synchronization.

The boundary of the region corresponding to the onset of phase synchronization on the plane of control parameters (ω_d, ε) for the system under consideration is shown in the figure. The appearance of phase synchronization was determined by the condition of entrainment of phases of the drive and response oscillators (3). The phases were defined as the polar angles in the coordinate systems on (x_d, y_d) and (x_r, y_r) planes. It should be noted that the boundary of the region of phase synchronization in cases when the phase was introduced in terms of the Hilbert transform or as the polar angle on the ($\dot{x}$, $\dot{y}$) plane virtually coincided with that presented in the figure, which is related to the good topology of the chaotic attractor in the system studied. As can be seen, the region of phase synchronization has the shape of a "tong," which touches the point (ω_r, 0) on the plane of control parameters (ω_d, ε) and then expands with increasing coupling parameter ε.

The same diagram shows a boundary corresponding to the onset of time scale synchronization for the base Morlet wavelet parameter $\Omega = 2\pi$. This choice (proposed in [8, 9, 11]) is determined by the fact that it provides for a simple and natural relationship between the time scale s and the frequency f of the corresponding Fourier spectrum: $s = 1/f$. For other values of Ω, relations between the frequency and the time scale are more involved (for detail, see [22]).

In the case of a relatively large mismatch between the parameters of interacting oscillators (see the figure), the regime of time scale synchronization is observed at lower values of ε as compared to that corresponding to the onset of phase synchronization. In the case of a small mismatch, the boundaries of the onset of both types of chaotic synchronization practically coincide.

This behavior is related to the resolving power of the wavelet function for a given value of Ω: as long as the detuning of frequencies of the drive and response oscillators is small, the synchronous behavior of both systems on the time scale s_1 (corresponding to the fundamental frequency in the Fourier spectrum of the drive system) can be "masked" by asynchronous dynamics on a time scale s_2 corresponding to the fundamental frequency of the response system [22]. For this reason, asynchronous behavior on both s_1 and s_2 scales will be observed unless the asynchronous dynamics on the s_2 scale is not completely suppressed by the synchronous dynamics on the s_1 scale, which corresponds to establishment of the regime of phase synchronization.

As the mismatch between the drive and response oscillators increases, the interval between the time scales s_1 and s_2 grows accordingly and the dynamics on these scales can be distinguished with the aid of the wavelet function. For this reason, the threshold of the onset of time scale synchronization for a large ω_d–ω_r difference is detected before the phase synchronization threshold.

An increase in the parameter Ω leads to an increase in the resolving power of the Morlet wavelet function (see, e.g., [22]), which makes possible the detection of fine details in the synchronous dynamics. Therefore, the threshold of the onset of time scale synchronization must shift toward lower values of the coupling parameter ε with increasing Ω and, on the contrary, a decrease in Ω will shift the threshold of detection of the regime of time scale synchronization toward greater ε values. Simultaneously, a decrease in Ω is accompanied by expansion of a region on the plane of control parameters (ω_d, ε) in which the boundaries of the time scale synchronization and the phase synchronization coincide. This is illustrated in the figure, which also shows a boundary of the time scale synchronization determined using the Morlet wavelet function with $\Omega = 16$. As can be seen, the time scale synchronization in this case is detected earlier than in the case of $\Omega = 2\pi$.

For comparison, the diagram also shows a boundary of the time scale synchronization determined using a Morlet wavelet function with a very poor resolving power, which corresponds to $\Omega = 0.5$. In this case, the boundary of the time scale synchronization is detected virtually simultaneously with the onset of phase synchronization even for a relatively large mismatch between the parameters of interacting oscillators. Thus, an analysis of the behavior of coupled chaotic oscillators with the aid of the Morlet wavelet with a poor resolving power (small Ω) makes possible the detection of the boundary of the time scale synchronization even for a relative large mismatch between the two coupled oscillators. In such cases, the phase coherence (or its absence) of the chaotic attractor is probably not as important. In particular, for a system of two coupled Rössler oscillators described in [21] with a coupling parameter of $\varepsilon = 0.05$, a synchronous regime can be detected using a Morlet wavelet with $\Omega = 0.5$ even despite the fact that the chaotic attractor is not phase-coherent (in [21], the synchronous regime was detected by passing to the $(\dot{x}, \dot{y})$ plane)).

In conclusion, we have studied a relationship between the phase synchronization of chaotic oscillators and their time scale synchronization and determined the role of selection of the Morlet wavelet parameter on the detection of synchronous regimes in a system of coupled oscillators.

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