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Chaotic Phase Synchronization Studied by Means of Continuous Wavelet Transform

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Abstract—A new approach to introduction of the phase of a chaotic signal is developed based on the continuous wavelet transform. The proposed method is applied to the study of phase synchronization of two chaotic dynamical systems with ill-defined phases. *© 2004 MAIK "Nauka/Interperiodica".*

The phase synchronization of systems occurring in the regime of dynamical chaos [1–3] is one of the most important phenomena studied by modern theory of nonlinear oscillations. The effect of chaotic phase synchronization was experimentally observed in radio frequency generators [4], lasers [5], electrochemical oscillators [6], heart rhythm [7], gas discharge [8], etc. (see also [3, 9, 10]). Investigations into this phenomenon are also very important from the standpoint of data transmission by means of deterministic chaos [11].

The phenomenon of phase synchronization is usually described and analyzed in terms of the phase $\phi(t)$ of a chaotic signal [1, 3, 9, 10]. The phase synchronization implies that phases of the chaotic signals become entrained, while their amplitudes remain uncorrelated and appear as chaotic. The phase entrainment leads to coincidence of the signal frequencies, which are determined as average rates of the phase variation $\langle \dot{\phi}(t) \rangle$.

There is no universal method for introducing the phase of a chaotic signal which would provide correct results for arbitrary dynamical systems. Several methods have been developed for "well-behaving" systems with a relatively simple topology of chaotic attractors. First, the phase $\phi(t)$ of a chaotic signal is frequently introduced as an angle in the polar coordinate system on the (x, y) plane [12]. Second, the phase of a chaotic system is defined by considering an analytical expression $\zeta(t) = x(t) + jH[x(t)] = A(t)\exp[j\phi(t)]$, where $H[x(t)]$ is the Hilbert transform of the chaotic signal $x(t)$ [13]. Third, the phase of a chaotic signal is defined on a surface of the Poincaré cross section and it is assumed that the signal phase exhibits linear variation in the interval between two sequential intersections of the phase trajectory with the Poincaré cross section [1, 3, 9]. All these approaches give analogous correct results for "well-behaving" systems [1, 3, 9, 10]. On the other hand, these methods frequently lead to incorrect results for the systems with ill-defined phases (see, e.g., [3, 15]). In such cases, the phase synchronization can be revealed by means of indirect measurements [3, 9, 14].

Below we propose a new method for establishing the phase synchronization of dynamical systems with ill-defined phases. The behavior of such a system can be described by introducing a continuous set of phases determined by means of the continuous wavelet transform [16, 17] of a chaotic signal $x(t)$:

$$W(s, t_0) = \int_{-\infty}^{+\infty} x(t)\psi^*_{s,t_0}(t)dt, \tag{1}$$

where $\psi_{s,t_0}(t)$ is the wavelet function (the asterisk denotes complex conjugation) obtained from the base wavelet $\psi_0(t)$

$$\psi_{s,t_0}(t) = \frac{1}{\sqrt{s}}\psi_0\left(\frac{t-t_0}{s}\right). \tag{2}$$

s is the analyzed time scale determining the width of the $\psi_{s,t_0}(t)$ wavelet; and t_0 is the shift of the wavelet function along the time axis. Note that, in the wavelet analysis, the concept of "time scale" is usually employed instead of the term "frequency" traditionally used in the Fourier transform analysis.

For the base wavelet, we use the Morlet wavelet $\psi_0(\eta) = (1/\sqrt[4]{\pi})\exp(j\omega_0\eta)\exp(-\eta^2/2)$ [16]. Selection of the wavelet parameter $\omega_0 = 2\pi$ provides for the relation $s \simeq 1/f$ between the time scale s of the wavelet transform and the frequency f of the Fourier transform.

The wavelet surface

$$W(s, t_0) = |W(s, t_0)|\exp[j\phi_s(t_0)] \tag{3}$$

describes behavior of the system for each time scale s at any moment of time t_0. The magnitude of $|W(s, t_0|$ characterizes the presence and intensity of the corre-

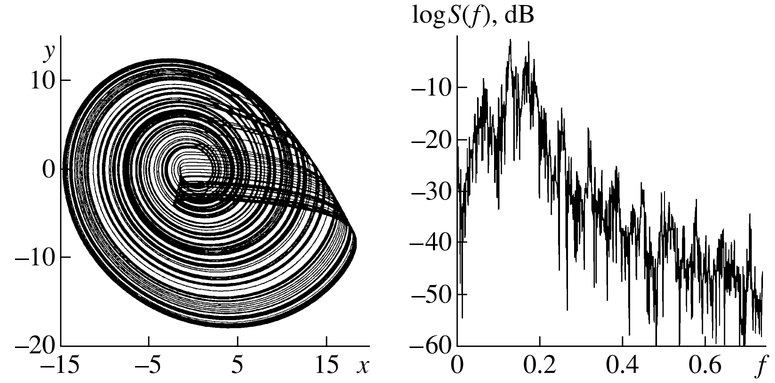

Fig. 1. The projection of a phase portrait on the (x, y) plane and the power spectrum of the first Rössler system (coupling parameter $\varepsilon = 0$).

sponding time scale s at the moment of time t_0. It is also convenient to introduce the integral distribution of the wavelet energy with respect to the time scales

$$E(s) = \int |W(s, t_0)|^2 dt_0.$$

The phase is naturally defined as $\phi_s(t) = \arg W(s, t)$ for each time scale s In other words, it is possible to characterize the behavior of each time scale s in terms of the associated phase $\phi_s(t)$.

Let us consider the behavior of two mutually coupled nonidentical chaotic oscillators. If these oscillators do not occur in the state of phase synchronization, their behavior is not synchronized for all time scales s. As soon as some of the time scales of these dynamical systems become synchronized (e.g., as a result of increase in the coupling parameter), the phase synchronization regime is established. Evidently, this synchronization primarily takes place for the time scales corresponding to the maximum energy fraction of the wavelet spectrum $E(s)$, while the other scales remain desynchronized. The phase synchronization leads to the phase entrainment on the synchronized time scales s such that

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \text{const}, \quad (4)$$

where $\phi_{s1,2}(t)$ are continuous phases of the first and second oscillators corresponding to the synchronized time scales s.

The proposed approach based on the continuous wavelet transform can be successfully applied to any dynamical systems, including those with ill-defined phases. For example, let us consider the behavior of two mutually coupled nonidentical Rössler systems occurring in the vortex chaos regime (Fig. 1):

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \quad (5)$$

where ε is the coupling parameter, $\omega_1 = 0.98$, $\omega_2 = 1.03$, $a = 0.22$, $p = 0.1$, and $c = 8.5$.

Figure 2 shows the results of calculations for two mutually coupled Rössler systems in the case of a small coupling parameter ($\varepsilon = 0.025$). The wavelet transform power spectra $E(s)$ of the two systems differ in shape (see Fig. 2a), but the maximum values of the energy ε correspond to approximately the same time scale s. According to Fig. 2a, the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ exhibits infinite growth for all time scales. This implies that no synchronized time scales exist in the systems under consideration and, hence, the systems exhibit no phase synchronization.

As the coupling parameter increases, the systems exhibit phase synchronization (see, e.g., [12]). The results of indirect measurements [14] showed that the coupled Rössler systems occur in the regime of phase synchronization in the case of $\varepsilon = 0.05$. Behavior of the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ in this case is illustrated by Fig. 2b. As can be seen, there is phase synchronization for the time scale $s = 5.25$ (characterized by a maximum energy of the wavelet spectrum $E(s)$ (Fig. 2b). Thus, the time scales $s = 5.25$ of the two Rössler systems are synchronized. Simultaneously, all the time scales in the nearest vicinity of $s = 5.25$ are synchronized as well. Strongly different time scales (e.g., $s = 4.5$ or 6.0) remain unsynchronized (see Fig. 2b in comparison to Fig. 2a).

Further increase in the coupling parameter (e.g., to $\varepsilon = 0.07$) leads to phase synchronization for the time scales where no such synchronization took place before. As can be seen from Fig. 2c, the time scales $s = 4.5$ in the two coupled systems are synchronized (in contrast to the case of $\varepsilon = 0.05$ illustrated in Fig. 2b). The interval of time scales featuring phase entrainment increases, but some time scales (e.g. $s = 3.0$ and 6.0 in Fig. 2c) remain desynchronized as before.

The above results reveal several important aspects. First, traditional approaches to determining phase synchronization based on the introduction of a chaotic sig-

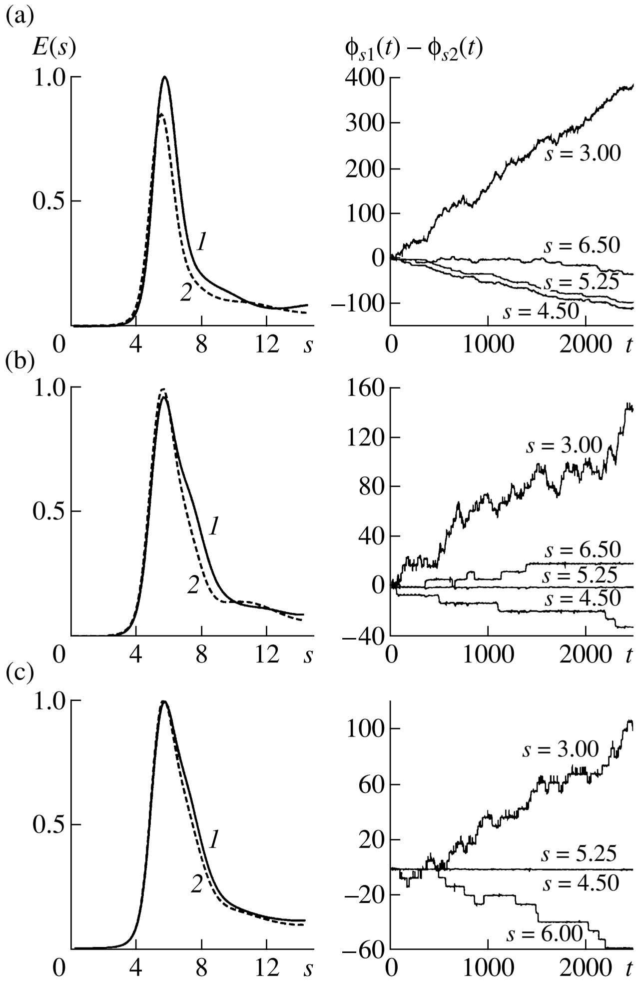

Fig. 2. Normalized energy spectra $E(s)$ of the wavelet transforms for the first (solid curves *1*) and second (dashed curves *2*) Rössler systems and behavior of the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ of these systems for various coupling parameter: (a) $\varepsilon = 0.025$ (no phase synchronization); (b) $\varepsilon = 0.05$ (time scales $s = 5.25$ are synchronized and the two systems exhibit phase synchronization); $\varepsilon = 0.07$ (the number of synchronized time scales and the interval of phase synchronization increase).

nal phase provide a correct description for time series characterized by the Fourier spectrum with a sharply pronounced main frequency f_0. In such cases, the phase ϕ_{s0} introduced for the time scale $s_0 \simeq 1/f_0$ approximately coincides with the chaotic signal phase $\phi(t)$ introduced in the classical way. Indeed, since the other frequencies (or the other time scales) are not pronounced in the Fourier spectrum, the phase $\phi(t)$ of the characteristic signal is close to $\phi_{s0}(t)$ for the main frequency f_0 (and, accordingly, for the main time scale s_0). Obviously, the average frequencies $\bar{f} = \langle \dot{\phi}(t) \rangle$ and $\bar{f}_{s0} = \langle \dot{\phi}_{s0}(t) \rangle$ must also coincide with each other and with the main frequency of the Fourier spectrum $\bar{f} = \bar{f}_{s0} = f_0$ (see also [15]).

If the given chaotic time series is characterized by a Fourier spectrum featuring no clearly pronounced main spectral component (as, e.g., in the spectrum of the Rössler system in Fig. 1), the traditional approaches to introduction of the chaotic signal phase (mentioned in the introduction) are no longer valid and may lead to incorrect results. In contrast, the proposed approach based on a continuous wavelet transform and introduction of a continuous phase set can be applied to description of a chaotic signal of any type.

The second important circumstance is that the proposed method requires no a priori information about the system studied and, hence, can be used for an analysis of experimental data. Moreover, use of the wavelet analysis sometimes decreases the influence of noise [16, 18]. It is quite possible that the method described above can be also useful and effective in the analysis of time series generated by physical, biological, physiological, and other systems.

Thus, we have proposed a new approach to description of the phenomenon of phase synchronization, which is based on the continuous wavelet transform and analysis of the system dynamics for various time scales. This approach can be used for the description of any chaotic systems (including those with ill-defined phases) and experimental time series.

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