An approach to chaotic synchronization

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(Received 20 April 2004; accepted 5 June 2004)

This paper deals with the chaotic oscillator synchronization. An approach to the synchronization of chaotic oscillators has been proposed. This approach is based on the analysis of different time scales in the time series generated by the coupled chaotic oscillators. It has been shown that complete synchronization, phase synchronization, lag synchronization, and generalized synchronization are the particular cases of the synchronized behavior called as "time–scale synchronization." The quantitative measure of chaotic oscillator synchronous behavior has been proposed. This approach has been applied for the coupled Rössler systems and two coupled Chua's circuits. [DOI: 10.1063/1.1775991]

Synchronization of chaotic oscillators is one of the fundamental phenomena of nonlinear dynamics. There are several different types of synchronization of coupled chaotic oscillators which have been described theoretically and observed experimentally. In this paper we propose an approach to the synchronization of two coupled chaotic oscillators based on the consideration of oscillators' time scale dynamics. We have shown that synchronized behavior of time scales should be considered as a type of synchronization called "*time scale synchronization*" and other types of chaotic synchronization are the particular cases of time scale synchronization.

I. INTRODUCTION

Synchronization of chaotic oscillators is one of the fundamental phenomena of nonlinear dynamics. It takes place in many physical[1–6] and biological[7–9] processes. It seems to play an important role in the ability of biological oscillators, such as neurons, to act cooperatively.[10–12] Chaotic synchronization can also be used for secret signal transmission.[13,14] There are several different types of synchronization of coupled chaotic oscillators which have been described theoretically and observed experimentally.[15–20] These are the *complete synchronization* (CS),[21,22] *lag synchronization* (LS),[23,24] *generalized synchronization* (GS),[25,26] and *phase synchronization* (PS).[15,19] In this paper we propose an approach to the synchronization of two coupled chaotic oscillators based on the consideration of oscillators' time scale dynamics. We have shown that synchronized behavior of time scales should be considered as a type of synchronization called "*time scale synchronization*" (TSS) and CS, LS, PS, and GS are the particular cases of TSS.

The *complete synchronization* (CS) implies coincidence of states of coupled oscillators $\mathbf{x}_1(t)\cong\mathbf{x}_2(t)$, the difference between state vectors of coupled systems converges to zero in the limit $t\to\infty$.[13,21,22,27] It appears when interacting systems are identical. If the parameters of coupled chaotic oscillators slightly mismathch, the state vectors are close $|\mathbf{x}_1(t)-\mathbf{x}_2(t)|\approx 0$, but differ from each other. Another type of synchronized behavior of coupled chaotic oscillators with slightly mismatched parameters is the *lag synchronization* (LS), when shifted in time, the state vectors coincide with each other, $\mathbf{x}_1(t+\tau)=\mathbf{x}_2(t)$. When the coupling between the oscillator increases the time lag τ decreases and the synchronization regime tends to be a CS one.[23,24,28] The *generalized synchronization* (GS)[25,26,29] introduced for drive-responce systems, means that there is some functional relation between coupled chaotic oscillators, i.e., $\mathbf{x}_2(t)=\mathbf{F}[\mathbf{x}_1(t)]$.

Finally, it is necessary to mention the *phase synchronization* (PS) regime. To describe the phase synchronization the instantaneous phase $\phi(t)$ of a chaotic continuous time series is usually introduced.[15,17–19,30,31] The phase synchronization means the entrainment of phases of chaotic signals, whereas their amplitudes remain chaotic and uncorrelated.

All synchronization types mentioned above are associated with each other (see, for detail, Refs. 1, 25, and 28), but the relationship between them is not completely clarified yet. For each type of synchronization there are their own ways to detect the synchronized behavior of coupled chaotic oscillators. The complete synchronization can be displayed by means of comparison of system state vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, whereas the lag synchronization can be determined by means of the similarity function.[23] The case of the generalized synchronization is more intricate because the functional relation $\mathbf{F}[\cdot]$ can be very complicated, but there are several methods to detect the synchronized behavior of coupled chaotic oscillators, such as the auxiliary system approach[32] or the method of nearest neighbors.[25,33]

Finally, the phase synchronization of two coupled chaotic oscillators occurs if the difference between the instantaneous phases $\phi_{1,2}(t)$ of chaotic signals $\mathbf{x}_{1,2}(t)$ is bounded by some constant

$$|\phi_1(t)-\phi_2(t)|<\text{const.} \quad (1)$$

It is possible to define a mean frequency of chaotic signal

$$\bar{\Omega}=\lim_{t\to\infty}\frac{\phi(t)}{t}=\langle\dot{\phi}(t)\rangle, \quad (2)$$

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1054-1500/2004/14(3)/1/8/$22.00

which should be the same for both coupled chaotic systems, i.e., the phase locking leads to the frequency entrainment. It is important to notice that to obtain correct results the mean frequency $\bar{\Omega}$ of chaotic signal $\mathbf{x}(t)$ should coincide with the main frequency $\Omega_0=2\pi f_0$ of the Fourier spectrum (for detail, see, Ref. 34).

In this paper we propose an approach to the synchronization of two coupled chaotic oscillators. The main idea of this approach consists in the analysis of the system behavior on different time scales that allows us to consider different cases of synchronization from the universal positions. The type of synchronous behavior [so-called time scale synchronization (TSS)] has been introduced.

The structure of this paper is as follows. In Sec. I we discuss the method of the time scales s and associated with them phases $\phi_s(t)$ definition by means of the continuous wavelet transform. The concept of time scale synchronization is given in Sec. II. Section III deals with the synchronization of two mutually coupled Rössler systems with funnel attractors. We demonstrate the efficiency of our method for such cases and discuss the correlation between PS, LS, CS, and TSS. Section IV deals with the time scale synchronization of two coupled chaotic Chua's circuits. In Sec. V we consider application of our method for the unidirectional coupled Rössler systems when the generalized synchronization is observed. The quantitative measure of synchronization is described in Sec. VI. The final conclusion is presented in Sec. VII.

II. CONTINUOUS WAVELET TRANSFORM AND TIME SCALES DYNAMICS

The continuous wavelet transform[35–38] is the powerful tool for the analysis of nonlinear dynamical system behavior. In particular, the continuous wavelet analysis has been used for the detection of synchronization of chaotic oscillations in the brain[39–41] and chaotic laser array.[42] It has also been used to detect the main frequency of the oscillations in nephron autoregulation.[43] We propose to analyze the dynamics of coupled chaotic oscillators using the consideration of system behavior on different time scales s and each of them is characterized by means of its own phase $\phi_s(t)$, respectively. So, in order to define *the continuous set of instantaneous phases* $\phi_s(t)$ the continuous wavelet transform is the convenient mathematical tool.

Let us consider continuous wavelet transform of chaotic time series $x(t)$

$$W(s,t_0)=\int_{-\infty}^{+\infty} x(t)\psi^*_{s,t_0}(t)dt, \tag{3}$$

where $\psi_{s,t_0}(t)$ is the wavelet-function related to the mother-wavelet $\psi_0(t)$ as

$$\psi_{s,t_0}(t)=\frac{1}{\sqrt{s}}\psi\left(\frac{t-t_0}{s}\right). \tag{4}$$

The time scale s corresponds to the width of the wavelet function $\psi_{s,t_0}(t)$, and t_0 is shift of wavelet along the time axis, the symbol "*" in Eq. (3) denotes complex conjugation. It should be noted that the time scale s is usually used instead of the frequency f of Fourier transformation and can be considered as the quantity inversed to it.

The Morlet-wavelet[44]

$$\psi_0(\eta)=\frac{1}{\sqrt[4]{\pi}}\exp(j\Omega_0\eta)\exp\left(\frac{-\eta^2}{2}\right) \tag{5}$$

has been used as a mother-wavelet function. The choice of parameter value $\Omega_0=2\pi$ provides the relation $s=1/f$ between the time scale s of wavelet transform and frequency f of Fourier transformation.

The wavelet surface

$$W(s,t_0)=|W(s,t_0)|e^{j\phi_s(t_0)} \tag{6}$$

describes the system's dynamics on every time scale s at the moment of time t_0. The value of $|W(s,t_0)|$ indicates the presence and intensity of the time scale s mode in the time series $x(t)$ at the moment of time t_0. It is possible to consider the quantity

$$\langle E(s)\rangle=\int |W(s,t_0)|^2 dt_0, \tag{7}$$

which is the integral energy distribution on time scales, respectively. At the same time, the phase $\phi_s(t)=\arg W(s,t)$ is naturally introduced for every time scale s. In other words, $\phi_s(t)$ is continuous function of time t and time scale s.

II. TIME SCALE SYNCHRONIZATION

Using the continuous wavelet transform we have introduced the continuous set of time scales s and associated with them instantaneous phases $\phi_s(t)$ (see Sec. I). It means that it is possible to describe the behavior of each time scale s by means of its own phase $\phi_s(t)$. Let us consider the dynamics of two coupled oscillators. If in the time series $\mathbf{x}_{1,2}(t)$ generated by these systems there is time scale range $s_m\leqslant s\leqslant s_b$ for time scales s from which the phase locking condition

$$|\phi_{s1}(t)-\phi_{s2}(t)|<\text{const} \tag{8}$$

is satisfied and the part of the wavelet spectrum energy being fallen on this range is not equal to zero

$$E_{snhr}=\int_{s_m}^{s_b}\langle E(s)\rangle ds>0, \tag{9}$$

we say that *time scale synchronization* (TSS) between oscillators takes place.

It is obvious that the classical synchronization of coupled periodical oscillators is equal to TSS because in this case all time scales are synchronized according to the time scale s, instantaneous phase $\phi_s(t)$, and TSS definitions. The case of chaotic oscillations is more complicated. Nevertheless, as we will show further, if two chaotic oscillators demonstrate any type of synchronized behavior mentioned above (CS, LS, PS, or GS), in the time series $\mathbf{x}_{1,2}(t)$ generated by these systems there are time scales s necessarily correlated with each other for which the phase locking condition (8) is satisfied. Therefore, the time scale synchronization is also realized. In other words, CS, LS, PS, and GS are the particular cases of the time scale synchronization. To detect time

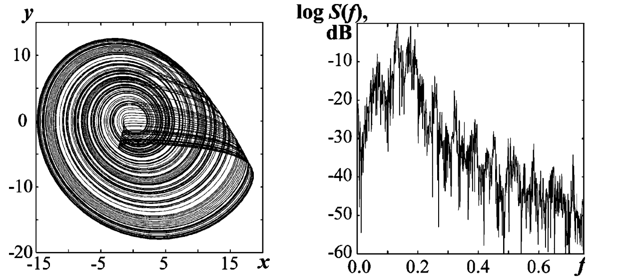

FIG. 1. Phase picture and power spectrum of the first Rössler system (10) oscillations. Coupling parameter ε is equal to zero.

scale synchronization one can examine the condition (8) which should be satisfied for synchronized time scales.

Let us consider several examples of coupled oscillators demonstrating different types of chaotic synchronization (PS, LS, GS). We will show that if any type of synchronous behavior is observed the time scale synchronization is detected, too. So, TSS is a general case of synchronization and CS, LS, PS, and GS are the particular cases of TSS.

III. EXAMPLE I. TIME SCALE SYNCHRONIZATION OF TWO RÖSSLER SYSTEMS: FROM PHASE TO LAG SYNCHRONIZATION

Let us start our consideration with two mutually coupled Rössler systems with slightly mismatched parameters. For this system it is impossible to correctly introduce the instantaneous phase $\phi(t)$ of chaotic signal $\mathbf{x}(t)$. It is clear that for such cases the traditional methods of the phase synchronization detecting fail and it is necessary to use the other techniques, e.g., indirect measurements.[45] On the contrary, our approach allows us to easily detect the time scale synchronization between chaotic oscillators.

To illustrate it we consider two nonidentical coupled Rössler systems with funnel attractors (Fig. 1)

$$\begin{aligned}\dot{x}_{1,2}&=-\omega_{1,2}y_{1,2}-z_{1,2}+\varepsilon(x_{2,1}-x_{1,2}),\\ \dot{y}_{1,2}&=\omega_{1,2}x_{1,2}+ay_{1,2}+\varepsilon(y_{2,1}-y_{1,2}),\\ \dot{z}_{1,2}&=p+z_{1,2}(x_{1,2}-c),\end{aligned}\tag{10}$$

where ε is a coupling parameter, $\omega_1=0.98$, $\omega_2=1.03$. The control parameter values have been selected by analogy with Ref. 45 as $a=0.22$, $p=0.1$, and $c=8.5$. It is necessary to note that under these control parameter values none of the methods mentioned above permits us to define the phase of chaotic signal correctly in the whole range of coupling parameter ε variation. Therefore, nobody can determine by means of direct measurements whether the phase synchronization regime takes place for several values of parameter ε. On the contrary, our approach allows us to detect TSS synchronization between considered coupled oscillators easily for all values of coupling parameter.

In Ref. 45 it has been shown by means of the indirect measurements that for the coupling parameter value $\varepsilon=0.05$ the synchronization of two mutually coupled Rössler systems (10) takes place. Our approach based on the analysis of the dynamics of different time scales s gives analogous results. So, the behavior of the phase difference $\phi_{s1}(t)-\phi_{s2}(t)$ for this case has been presented in Fig. 2(b). One can see that the phase locking takes place for the time scales $s=5.25$ which are characterized by the largest energy value in the wavelet power spectra $\langle E(s)\rangle$ [see Fig. 2(b)].

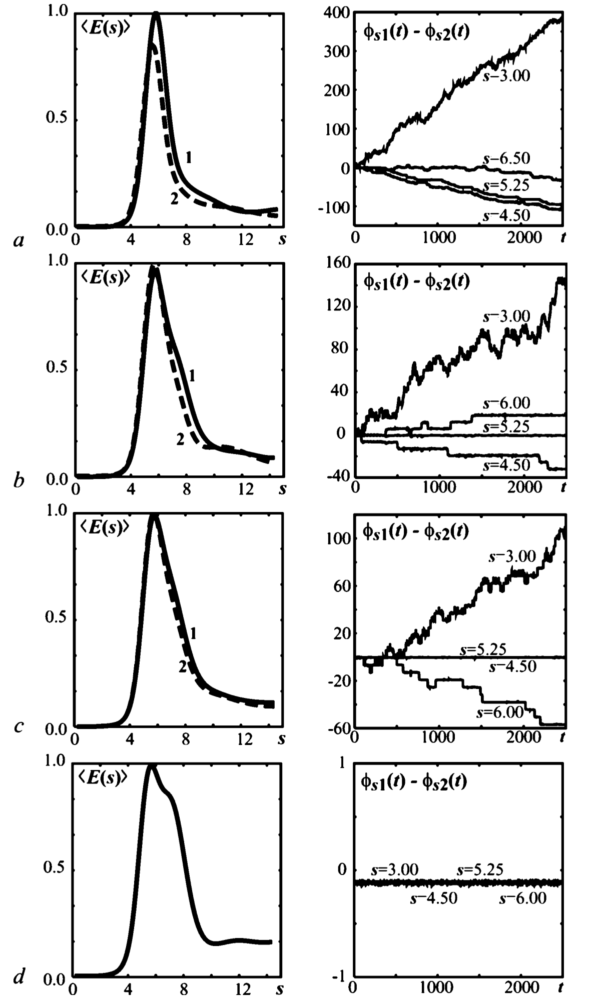

FIG. 2. The normalized energy distribution in wavelet spectrum $\langle E(s)\rangle$ for the first (the solid line denoted "1") and the second (the dashed line denoted "2") Rössler systems (10). The phase difference $\phi_{s1}(t)-\phi_{s2}(t)$ for two coupled Rössler systems. (a) The value of the coupling parameter has been selected as $\varepsilon=0.025$. There is no phase synchronization between systems; (b) the value of the coupling parameter has been selected as $\varepsilon=0.05$. The time scales $s=5.25$ are correlated with each other and the synchronization has been observed; (c) the value of the coupling parameter has been selected as $\varepsilon=0.07$; and (d) the value of the coupling parameter has been selected as $\varepsilon=0.25$. The lag synchronization has been observed, all time scales are synchronized.

It is important to note that the phase difference $\phi_{s1}(t)-\phi_{s2}(t)$ is also bounded on the time scales close to $s=5.25$. So, we can say that the time scales $s=5.25$ (and close to them) of two oscillators are synchronized with each other. At the same time the other time scales (e.g., $s=4.5$, 6.0, *et al.*) remain uncorrelated. For such time scales the phase locking has not been observed [see Fig. 2(b)].

It is clear that the synchronization phenomenon is caused by the existence of time scales s in system dynamics correlated with each other. It has been shown in Ref. 23 that there is certain relationship between PS, LS, and CS for chaotic oscillators with slightly mismatched parameters. With the increase of coupling strength the systems undergo the transition from unsynchronized chaotic oscillations to the phase synchronization. With a further increase of coupling the lag synchronization is observed. Further increasing of the coupling parameter leads to the decreasing of the time lag and both systems tend to have the complete synchronization regime.

Let us consider the dynamics of different time scales s of two nonidentical mutually coupled Rössler systems (10) when the coupling parameter value increases. If there is no phase synchronization between the oscillators, then their dynamics remain uncorrelated for all time scales s. Figure 2(a) illustrates the dynamics of two coupled Rössler systems when the coupling parameter ε is small enough ($\varepsilon = 0.025$). The power spectra $\langle E(s) \rangle$ of wavelet transform for Rössler systems differ from each other [Fig. 2(a)], but the maximum values of the energy correspond approximately to the same time scale s in both systems. It is clear, that the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ is not bounded for almost all time scales [see Fig. 2(a)]. One can see that the phase difference $\varphi_{s1}(t) - \varphi_{s2}(t)$ increases for time scale $s = 3.0$, but decreases for $s = 4.5$. It means that there should be the time scale $3 < s^* < 4.5$ the phase difference on which remains bounded. This time scale s^* plays a role of a point separating the time scale areas with the phase difference increasing and decreasing, respectively. In this case the measure of time scales on which the phase difference remains bounded is zero [therefore, the synchronized energy of wavelet power spectra (9) is equal to zero] and we can not say about the synchronous behavior of coupled chaotic oscillators (see also Sec. VI).

As soon as any of the time scales of the first chaotic oscillator becomes correlated with the other one of the second oscillator (e.g., when the coupling parameter increases), the phase synchronization occurs [see Fig. 2(b)]. The time scales s characterized by the largest value of energy in wavelet spectrum $\langle E(s) \rangle$ are more likely to become correlated first. The other time scales remain uncorrelated as before. The phase synchronization between chaotic oscillators leads to the phase locking (8) on the correlated time scales s.

When the parameter of coupling between chaotic oscillators increases, more and more time scales become correlated and one can say that the degree of the synchronization grows. So, with the further increasing of the coupling parameter value (e.g., $\varepsilon = 0.07$) in the coupled Rössler systems (10) the time scales which were uncorrelated before become synchronized [see Fig. 2(c)]. It is evident, that the time scales $s = 4.5$ are synchronized in comparison with the previous case [$\varepsilon = 0.05$, Fig. 2(b)] when these time scales were uncorrelated. The number of time scales s demonstrating the phase locking increases, but there are nonsynchronized time scales as before (e.g., the time scales $s = 3.0$ and $s = 6.0$ remain still nonsynchronized).

With further coupling parameter increasing the regime of lag synchronization[23] appears. Before the LS regime occurs the interesting phenomenon called as *the intermittent lag synchronization* (ILS) may be observed (see, for detail, Ref. 46). It means that most of the time the system demonstrates the behavior like lag synchronization, but there are bursts of local nonsynchronous behavior. In this case most of time scales are synchronized but there are time scales the phase locking condition (8) on which does not satisfied.

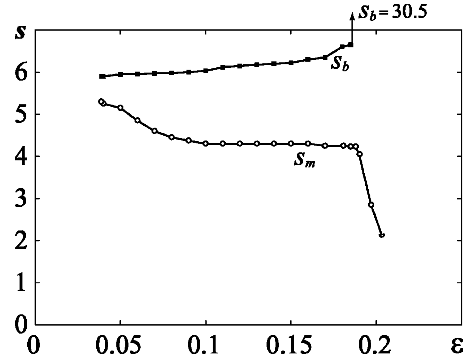

FIG. 3. The dependence of the synchronized time scale range $[s_m ; s_b]$ on the coupling strength ε for two mutually coupled Rössler systems (10) with funnel attractors.

An arising of the lag synchronization between oscillators means that all time scales are correlated. Indeed, from the condition of the lag–synchronization $x_1(t-\tau) \simeq x_2(t)$ one can obtain that $W_1(s,t-\tau) \simeq W_2(t,s)$ and therefore $\phi_{s1}(t-\tau) \simeq \phi_{s2}(t)$. It is clear, in this case the phase locking condition (8) is satisfied for all time scales s. For instance, when the coupling parameter of chaotic oscillators (10) becomes large enough ($s = 0.25$) the lag synchronization of two coupled oscillators appears. In this case the power spectra of wavelet transform coincide with each other [see Fig. 2(d)] and the phase locking takes place for all time scales s [Fig. 2(d)]. It is important to note that the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ is not equal to zero for the case of the lag synchronization. It is clear that this difference depends on the time lag τ.

Further increasing of the coupling parameter leads to the decreasing of the time lag τ.[23] Both systems tend to have the complete synchronization regime $x_1(t) \simeq x_2(t)$, so the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ tends to be a zero for all time scales.

The dependence of synchronized time scale range $[s_m ; s_b]$ on coupling parameter ε has been shown in Fig. 3. The range $[s_m ; s_b]$ of synchronized time scales appears at $\varepsilon \approx 0.039$. The appearance of synchronized time scale range corresponds to the phase synchronization regime. When the coupling parameter value increases the range of synchronized time scales expands until all time scales become synchronized. Synchronization of all time scales means the presence of lag synchronization regime.

So, we can say the time scale synchronization (TSS) is the most general synchronization type uniting (at least) PS, ILS, LS, and CS regimes. The regime of lag synchronization and the phase synchronization differ from each other only in the number of synchronized time scales.

IV. EXAMPLE II. TIME SCALE SYNCHRONIZATION OF CHUA'S CIRCUITS

Let us consider the dynamics of two mutually coupled Chua's circuits[47,48] with piecewise linear characteristic (see Fig. 4). The important feature of Chua's circuit is the pres-

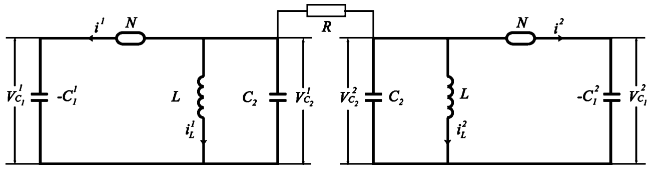

FIG. 4. Circuit realization of two mutually coupled Chua's oscillators.

ence of two characteristic time scales s_1 and s_2 (or two characteristic frequencies f_1 and f_2). It makes possible the realization in this system both quasiperiodic and chaotic oscillations.

The behavior of two coupled oscillators is described by

$$\dot{x}_{1,2} = -\frac{\alpha_{1,2}}{\gamma} f(y_{1,2} - x_{1,2}),$$

$$\dot{y}_{1,2} = -\frac{1}{\gamma}(f(y_{1,2} - x_{1,2}) + z_{1,2}) + \frac{\varepsilon}{\gamma}(y_{2,1} - y_{1,2}), \tag{11}$$

$$\dot{z}_{1,2} = \gamma y_{1,2},$$

where $x_{1,2} = V_{C_1}^{1,2}/E$ and $y_{1,2} = V_{C_2}^{1,2}/E$ are dimensionless voltages on capacitors $C_1^{1,2}$ and C_2 of the first and the second oscillators, respectively. The variable $z_{1,2} = i_L^{1,2}/I$ is the dimensionless current. The parameters E and I are the normalization factors. Dimensionless control parameters are $\alpha_{1,2} = C_2/C_1^{1,2}$ and $\gamma = 1/m_1\sqrt{C_2/L}$; $\tau = t/\sqrt{LC_2}$ is dimensionless time. The function

$$f(\xi) = -\frac{m_0}{m_1}\xi + \frac{1}{2}\left(\frac{m_0}{m_1}\right)(|\xi+1| - |\xi-1|), \tag{12}$$

is the dimensionless voltage-current characteristic of nonlinear element N, where m_0 and m_1 are the conductivities of the corresponding branches of voltage-current characteristic. The coupling parameter $\varepsilon = 1/Rm_1$ determines the influence of coupled Chua's circuits on each other.

The control parameter values have been selected as $\alpha_1 = 2.78$, $\alpha_2 = 2.89$, and $\gamma = 3.00$. The chaotic attractor and fourier spectrum of the first Chua's circuit oscillations is shown in Fig. 5, the characteristic frequencies have been denoted as $f_1 \simeq 0.161$ and $f_2 \simeq 0.032$. The dynamics of the second Chua's circuit is quite similar. Therefore, one can see in wavelet power spectra $E_{1,2}(s)$ two maxima on time scales $s_1 \simeq 6.2$ and $s_2 \simeq 30.0$ corresponding to the frequencies f_1 and f_2, respectively.

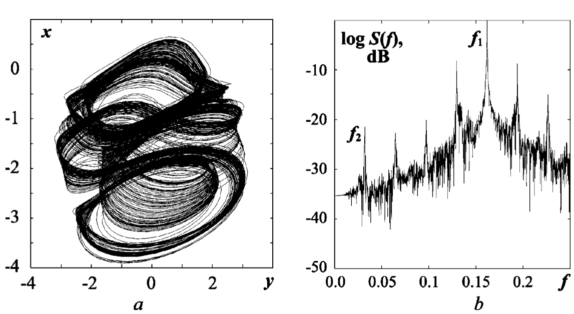

FIG. 5. The chaotic attractor and fourier spectrum of the first Chua's circuit. The coupling parameter ε is equal to zero.

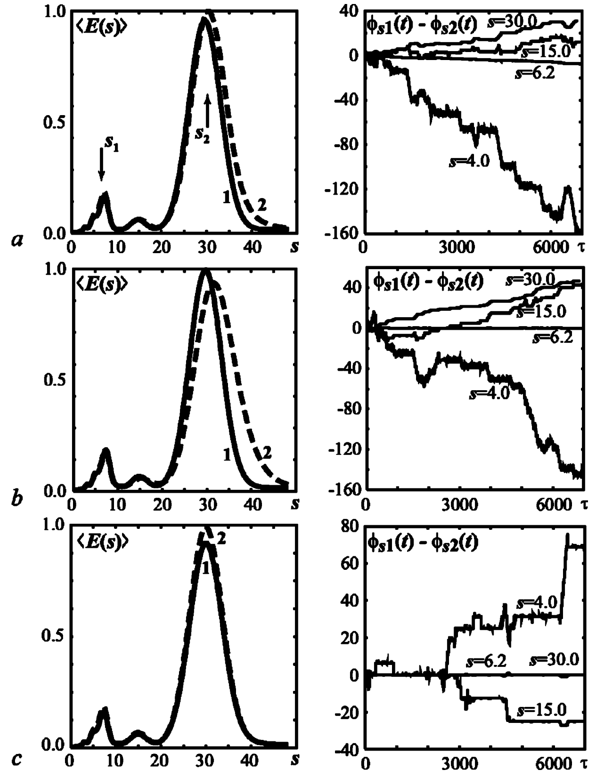

FIG. 6. Wavelet power spectra of time series $x_{1,2}(t)$ of two coupled Chua's circuit (11) (solid line "1" corresponds to the first Chua's circuit, dashed line corresponds to the second circuit, respectively) and phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ on different time scales s. The value of the coupling parameter ε is (a) $\varepsilon = 0.0$, (b) $\varepsilon = 0.05$, and (c) $\varepsilon = 0.25$,

When the coupling parameter is small TSS is not observed at all [Fig. 6(a)]. When the parameter ε increases the TSS appears on the time scales s_1 and close to them [see Fig. 6(b)]. Synchronous behavior on these time scales may be also detected as phase synchronization of coupled chaotic systems (11) by means of traditional approaches discussed in Refs. 15, 17–19, 30, 31. With further coupling parameter increasing, the second time scales s_2 become also synchronized. For these time scales the phase locking condition (8) and condition for wavelet spectra energy (9) are satisfied [see Fig. 6(c)]. It is important to note, that the appearance of synchronized behavior on time scales s_2 (and close to them) cannot be detected by means of traditional approaches as easily as before in the case of synchronous behavior on time scales s_1. In this case the synchronization on time scales s_2 is masked by synchronous behavior on time scales s_1.

So, the TSS allows to us analyze the chaotic behavior of the coupled systems with several spectral basic components in fourier spectrum. It is important to note that the synchronization phenomena can take place on the several different time scale ranges. In this case the energy being fallen on the synchronous time scales should be calculated as

$$E_{snhr} = \int_{s_{1m}}^{s_{1b}} \langle E(s) \rangle ds + \int_{s_{2m}}^{s_{2b}} \langle E(s) \rangle ds. \tag{13}$$

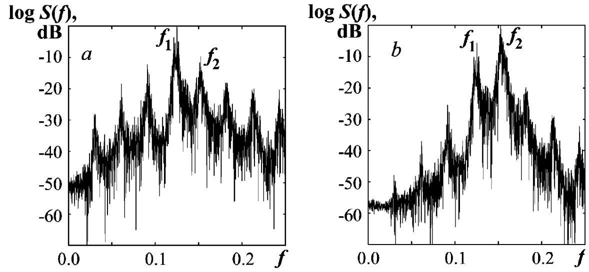

FIG. 7. Fourier spectra for (a) the first (drive) and (b) the second (response) Rössler systems (14). The coupling parameter is $\varepsilon=0.2$. The generalized synchronization takes place.

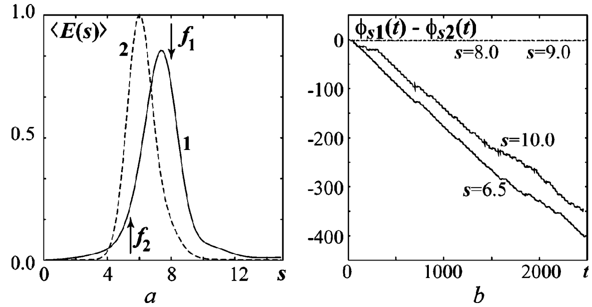

FIG. 8. (a) The normalized energy distribution in wavelet spectrum $\langle E(s)\rangle$ for the first (the solid line denoted "1") and the second (the dashed line denoted "2") Rössler systems. The time scales pointed with arrows correspond to the frequencies $f_1=0.125$ and $f_2=0.154$, respectively; (b) the phase difference $\phi_{s1}(t)-\phi_{s2}(t)$ for two coupled Rössler systems. The generalized synchronization has been observed.

V. EXAMPLE III. GENERALIZED SYNCHRONIZATION VERSUS TSS

Let us consider now another type of synchronized behavior, the so-called generalized synchronization. It has been shown above, that PS, ILS, LS, and CS are naturally interrelated with each other and the synchronization type depends on the number of synchronized time scales. The details of the relations between PS and GS is not at all clear. There are several works[1,28] dealing with the problem, how GS and PS are correlated with each other. For instance, in Ref. 28 it has been reported that two unidirectional coupled Rössler systems can demonstrate the generalized synchronization while the phase synchronization has not been observed. This case allows to be explained easily by means of the time scale analysis. The equations of Rössler system are

$$\begin{aligned}
\dot{x}_1&=-\omega_1 y_1-z_1,\\
\dot{y}_1&=\omega_1 x_1+a y_1,\\
\dot{z}_1&=p+z_1(x_1-c),\\
\dot{x}_2&=-\omega_2 y_2-z_2+\varepsilon(x_1-x_2),\\
\dot{y}_2&=\omega_2 x_2+a y_2,\\
\dot{z}_2&=p+z_2(x_2-c),
\end{aligned}\qquad (14)$$

where $\mathbf{x}_1=(x_1,y_1,z_1)^T$ and $\mathbf{x}_2=(x_2,y_2,z_2)^T$ are the state vectors of the first (drive) and the second (response) Rössler systems, respectively. The control parameter values have been chosen as $\omega_1=0.8$, $\omega_2=1.0$, $a=0.15$, $p=0.2$, $c=10$, and $\varepsilon=0.2$. The generalized synchronization takes place in this case (see Ref. 28 for detail). Why it is impossible to detect the phase synchronization in the system (14) despite the generalized synchronization is observed becomes clear from the time scale analysis.

Let us consider Fourier spectra of coupled chaotic oscillators (see Fig. 7). There are two main spectral components with frequencies $f_1=0.125$ and $f_2=0.154$ in these spectra. The analysis of behavior of time scales has shown that both the time scales $s_1=1/f_1=8$ of coupled oscillators corresponding to the frequency f_1 and time scales close to s_1 are synchronized while the time scales $s_2=1/f_2\simeq 6.5$ and close to them do not demonstrate synchronous behavior [Fig. 8(b)].

The source of such behavior of time scales becomes clear from the wavelet power spectra $\langle E(s)\rangle$ of both systems [see Fig. 8(a)]. The time scale s_1 of the drive Rössler system is characterized by the large value of energy while the part of energy being fallen on this scale of the response system is quite small. Therefore, the drive system dictates its own dynamics on the time scale s_1 to the response system. The opposite situation takes place for the time scales s_2 [see Fig. 8(a)]. The drive system cannot dictate its dynamics to the response system because the part of energy being fallen on this time scale is small in the first Rössler system and large enough in the second one. So, time scales s_2 are not synchronized.

Thus, the generalized synchronization of the unidirectional coupled Rössler systems appears as the time scale synchronized dynamics, as another synchronization types does before. It is also clear why the phase synchronization has not been observed in this case. The instantaneous phases $\phi_{1,2}(t)$ of chaotic signals $\mathbf{x}_{1,2}(t)$ introduced by means of traditional approaches are determined by both frequencies f_1 and f_2, but only the spectral components with the frequency f_1 are synchronized. So, the observation of instantaneous phases $\phi_{1,2}(t)$ does not allow us to detect the phase synchronization in this case although the synchronization of time scales takes place.

With increasing the coupling parameter between systems the range of synchronized time scales increases (Fig. 9) as

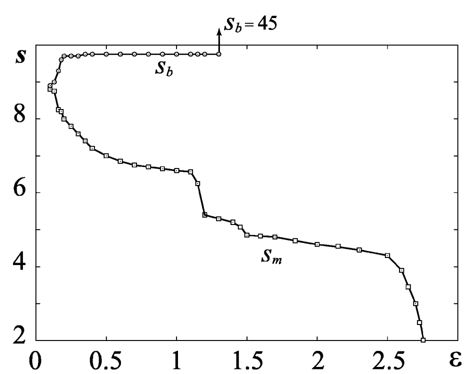

FIG. 9. The dependence of the synchronized time scale range $[s_m;s_b]$ on the coupling strength ε for two unidirectionally coupled Rössler systems (14).

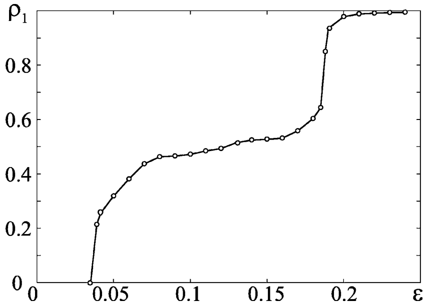

FIG. 10. The dependence of the synchronization measure ρ_1 for the first Rössler system (10) on the coupling strength ε. The measure ρ_2 for the second Rössler oscillator behaves in a similar manner, so it has not been shown in the figure.

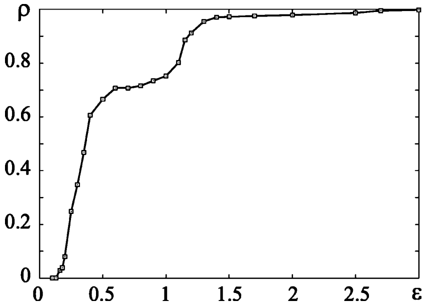

FIG. 11. The dependence of the synchronization measure ρ for the second Rössler system (14) on the coupling strength ε.

well as in the case of phase synchronization (see Sec. III). When all time scales become synchronized the lag synchronization appears.

Thus, one can see that there is a close relationship between different types of the chaotic oscillator synchronization. According to results mentioned above we can say that PS, LS, CS, and GS are particular cases of TSS. Therefore, it is possible to consider different types of synchronized behavior from the universal position. Unfortunately, it is not clear how one can distinguish the phase synchronization and the generalized synchronization using only the results obtained from the analysis of the time scale dynamics. [Here we mean the phase synchronization between chaotic oscillators takes place if the instantaneous phase $\phi(t)$ of chaotic signal may be correctly introduced by means of traditional approaches and the phase locking condition (1) is satisfied.]

VI. MEASURE OF SYNCHRONIZATION

From examples mentioned above one can see that any type of synchronous behavior of coupled chaotic oscillators leads to arising of the synchronized time scales. Therefore, the measure of synchronization can be introduced. This measure ρ can be defined as the the part of wavelet spectrum energy being fallen on the synchronized time scales

$$\rho_{1,2}=\frac{1}{E_{1,2}}\int_{s_m}^{s_b}\langle E_{1,2}(s)\rangle ds, \tag{15}$$

where $[s_m;s_b]$ is the range of time scales for which the condition (1) is satisfied and $E_{1,2}$ is a full energy of the wavelet spectrum

$$E_{1,2}=\int_{0}^{+\infty}\langle E_{1,2}(s)\rangle ds. \tag{16}$$

This measure ρ is 0 for the nonsynchronized oscillations and 1 for the case of complete and lag synchronization regimes. If the phase synchronization regime is observed it takes a value between 0 and 1 depending on the part of energy being fallen on the synchronized time scales. So, the synchronization measure ρ allows not only to distinguish the synchronized and nonsynchronized oscillations, but characterize quantitatively the degree of TSS synchronization.

Figure 10 presents the dependence of the TSS synchronization measure ρ_1 for the first Rössler oscillator of system (10) considered in Sec. III on the coupling parameter ε. It is clear that the part of the energy being fallen on the synchronized time scales grows monotonically with the growth of the coupling strength. Similar results (Fig. 11) have been obtained for the generalized synchronization of two coupled Rössler systems considered in Sec. V.

It has already mentioned above that when the coupled oscillators do not demonstrate synchronous behavior there are time scales s^* the phase difference $\phi_{s1}(t)-\phi_{s2}(t)$ on which is bounded. Such time scales play the role of points separating the time scale areas where the phase difference increases and decreases, respectively (see also Sec. III). Nevertheless, the presence of such time scales does not mean the occurrence of chaotic synchronization because the part of energy being fallen on them is equal to zero. Therefore, the synchronization measure ρ of such oscillations is zero, and dynamical regime being realized in the system in this case should be classified as nonsynchronous.

VII. CONCLUSION

Summarizing this work we would like to note several principal aspects. First, we have proposed to consider the time scale dynamics of coupled chaotic oscillators. It allows us to consider the different types of behavior of coupled oscillators (such as the complete synchronization, the lag synchronization, the phase synchronization, the generalized synchronization, and the nonsynchronized oscillations) from the universal position. In this case TSS is the most common type of synchronous coupled chaotic oscillator behavior. Therefore, the other types of synchronous oscillations (PS, LS, CS, ILS, and GS) may be considered as the particular cases of TSS. The quantitative characteristic ρ of the synchronization measure has also been introduced. It is important to note that our method (with insignificant modifications) can also be applied to dynamical systems synchronized by the external (e.g., harmonic) signal.

It is important to notice that this result agrees well with the idea proposed in Ref. 49 and developed in Refs. 16 and 50 that different kinds of chaotic synchronization might be captured in a single formalism. According to these works and results described in our paper one can see that there is an unifying framework for chaotic synchronization of coupled dynamical systems.

Second, the traditional approach for the phase synchronization detecting based on the introducing of the instantaneous phase $\phi(t)$ of chaotic signal is suitable and correct for such time series characterized by the Fourier spectrum with the single main frequency f_0. In this case the phase ϕ_{s0} associated with the time scale s_0 corresponding to the main frequency f_0 of the Fourier spectrum coincides approximately with the instantaneous phase $\phi(t)$ of the chaotic signal introduced by means of the traditional approaches (see also Ref. 51). Indeed, as the other frequencies (the other time scales) do not play a significant role in the Fourier spectrum, the phase $\phi(t)$ of the chaotic signal is close to the phase $\phi_{s0}(t)$ of the main spectral frequency f_0 (and the main time scale s_0, respectively). It is obvious, that in this case the mean frequencies $\bar{f}=\langle\dot{\phi}(t)\rangle/2\pi$ and $\bar{f}_{s0}=\langle\dot{\phi}_{s0}(t)\rangle/2\pi$ should coincide with each other and with the main frequency f_0 of the Fourier spectrum (see, also, Ref. 34)

$$\bar{f}=\bar{f}_{s0}=f_0. \tag{17}$$

If the chaotic time series is characterized by the Fourier spectrum without the main single frequency [like the spectrum shown in the Fig. 1(b)] the traditional approaches fail. One has to consider the dynamics of the system on all time scales, but it is impossible to do it by means of the instantaneous phase $\phi(t)$. On the contrary, our approach based on the time scale dynamics analysis can be used for both types of chaotic signals.

Finally, our approach can be easily applied to the experimental data because it does not require any *a priori* information about the considered dynamical systems. Moreover, in several cases the influence of the noise can be reduced by means of the wavelet transform (for detail, see, Refs. 35, 52, 53). We believe that our approach will be useful and effective for the analysis of physical, biological, physiological, and other data, such as Refs. 10, 39, 51.

ACKNOWLEDGMENTS

The authors express our our appreciation to George A. Okrokvertskhov and Alexander V. Kraskov and Professor Vadim S. Anishchenko and Professor Tatyana E. Vadivasova for valuable discussions. We also thank Svetlana V. Eremina for the support. This work has been supported by the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF), Grant No. REC-006. The authors also thank the ''Dynastiya'' Foundation.

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