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Generalized Synchronization of Chaotic Oscillators as a Partial Case of Time Scale Synchronization

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Abstract—Two coupled dynamical systems of the Rössler type are studied and it is shown that the generalized synchronization regime can be considered as a partial case of the time scale synchronization process. © *2004 MAIK "Nauka/Interperiodica".*

The process of chaotic synchronization of dynamical systems has been extensively studied in recent years [1–3]. This phenomenon is both of basic importance and of considerable practical interest (in particular, in biology [4, 5], data transfer by means of deterministic chaotic oscillations [6], etc.). According to modern classification, there are several types of chaotic synchronization, including generalized [7, 8], phase [9, 10], lag [11], and complete [12] synchronization. Recently [13, 14], it was shown that the phase, lag, and complete synchronization regimes are closely related, being essentially the manifestations of the same type of synchronous dynamics of the time scale of coupled chaotic oscillators, whereby the character of synchronization (phase, lag, or complete) is determined by the number of synchronized time scales introduced by means of a continuous wavelet transform [15].

In this Letter, we show that the regime of generalized synchronization is also a particular case of the synchronous behavior of time scales of a system of coupled chaotic oscillators.

The notion of generalized synchronization [7, 8] introduced for unidirectionally coupled dynamical systems, implies that there exists a certain function $\mathbf{F}[\cdot]$ relating the states of chaotic oscillators such that $\mathbf{x}_2(t) = \mathbf{F}[\mathbf{x}_1(t)]$. This functional relation can be very complicated, but there are methods capable of detecting the phenomenon of synchronization between unidirectionally coupled chaotic oscillators (see, e.g., [7, 16, 17]).

The time scale s and the associated phase $\phi_s(t)$ of a chaotic signal $\mathbf{x}(t)$ are defined by means of the continuous wavelet transform

$$W(s, t_0) = \int_{-\infty}^{+\infty} x(t)\psi^*_{s,t_0}(t)dt, \quad (1)$$

where $\psi_{s,t_0}(t)$ is the wavelet function (the asterisk denotes complex conjugation) obtained from the base wavelet $\psi_0(t)$

$$\psi_{s,t_0}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-t_0}{s}\right). \quad (2)$$

The time scale s determines the width of the $\psi_{s,t_0}(t)$ wavelet, while the parameter t_0 is the shift of the wavelet function along the time axis. For the base wavelet, we use the Morlet wavelet

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}}\exp(j\Omega_0\eta)\exp\left(\frac{-\eta^2}{2}\right). \quad (3)$$

Selection of the wavelet parameter $\Omega_0 = 2\pi$ provides for the relation $s \sim 1/f$ between the time scale s of the wavelet transform and the frequency f of the Fourier transform. It is also convenient to introduce the integral distribution of the wavelet energy with respect to the time scales, which is defined as

$$\langle E(s)\rangle = \int |W(s, t_0)|^2 dt_0. \quad (4)$$

Let $\mathbf{x}_{1,2}(t)$ be the time series generated by two coupled chaotic oscillators. If there exists a certain interval of time scales $[s_m; s_b]$ such that, for any time scale $s \in [s_m; s_b]$, the condition of phase entrainment

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \text{const} \quad (5)$$

is satisfied and the energy fraction of the wavelet spectrum in this interval is nonzero,

$$E_{snhr} = \int_{s_m}^{s_b} \langle E(s)\rangle ds > 0, \quad (6)$$

then the time scales $s \in [s_m; s_b]$ are synchronized and the chaotic oscillators occur in the regime of time scale synchronization.

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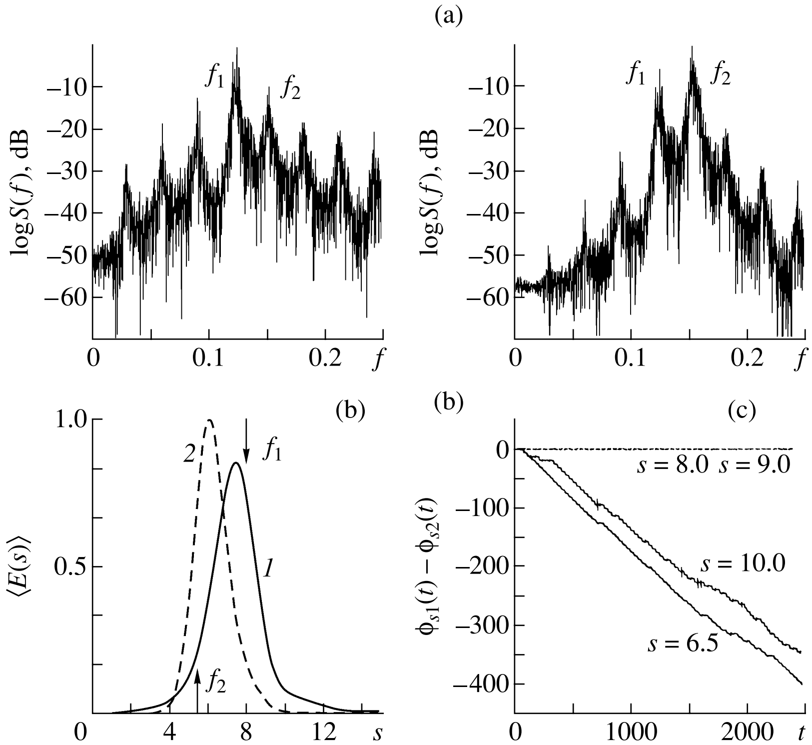

Fig. 1. Coupled chaotic Rössler systems (7): (a) logarithmic Fourier spectra log$S(f)$ of the master (left) and slave (right) systems with a coupling parameter of $\varepsilon = 0.2$; (b) normalized energy distribution $\langle E(s)\rangle$ in the wavelet spectrum of the master (*1*) and slave (*2*) system; arrows indicate the time scales corresponding to the main frequencies $f_1 = 0125$ and $f_2 = 0.154$; (c) phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for various time scales. Plots in (a, b) correspond to the regime of generalized synchronization.

In order to analyze this situation in more detail for two unidirectionally coupled oscillators occurring in the regime of generalized synchronization, let us consider two coupled Rössler systems,

$$\begin{aligned}
\dot{x}_1 &= -\omega_1 y_1 - z_1,\\
\dot{y}_1 &= \omega_1 r_1 + a y_1,\\
\dot{z}_1 &= p + z_1(x_1 - c),\\
\dot{x}_2 &= \omega_2 y_2 - z_2 + \varepsilon(x_1 - x_2),\\
\dot{y}_2 &= \omega_2 x_2 + a y_2\\
\dot{z}_2 &= p + z_2(x_2 - c),
\end{aligned} \tag{7}$$

where $\mathbf{x}_1 = (x_1, y_1, z_1)^T$ and $\mathbf{x}_2 = (x_2, y_2, z_2)^T$ are the state vectors of the first (master) and second (slave) systems, respectively, and ε is the coupling parameter. The values of the control parameters are selected as follows: $\omega_1 = 0.8$, $\omega_2 = 1.0$, $a = 0.15$, $p = 0.2$, $c = 10$, and $\varepsilon = 0.2$. It is known that two coupled Rössler systems with such parameters occur in the regime of generalized synchronization (for more detail, see [8]), while the phase synchronization does not take place.

How can it be that the phase synchronization regime is not established despite the fact that the generalized synchronization takes place, becomes clear from an analysis of the behavior of time scales. Figure 1a shows the Fourier spectra of the coupled chaotic oscillators. As can be seen, the spectra contain the main spectral components with the frequencies $f_1 = 0125$ and $f_2 = 0.154$. The analysis of the behavior of time scales shows that the time scales $s_1 = 1/f_1 = 8$ of the coupled oscillators corresponding to the frequency f_1 (and the scales close to s_1) are synchronized, whereas the time scales $s_2 = 1/f_2 \simeq 6.5$ corresponding to the frequency f_2 (and the scales close to s_2) exhibit nonsynchronous behavior (Fig. 1b). Since both these frequencies present in the spectrum influence the instantaneous phase of the chaotic signal introduced in one or another way (for more detail, see [9]), while the time scales corresponding to f_2 are not synchronized, the phenomenon of phase synchronization is not observed (see also [19]).

The reason for such a behavior of the time scales becomes clear from an analysis of the wavelet spectra $\langle E(s)\rangle$ of both Rössler systems (Fig. 1). Indeed, the time scale of the master system is characterized by a large energy, wile the fraction of energy corresponding to this time scale in the spectrum of the slave system is rather small. Fore this reason, the master system induces its dynamics in the slave system over the time scale s_1. On the contrary, the fraction of energy corre-

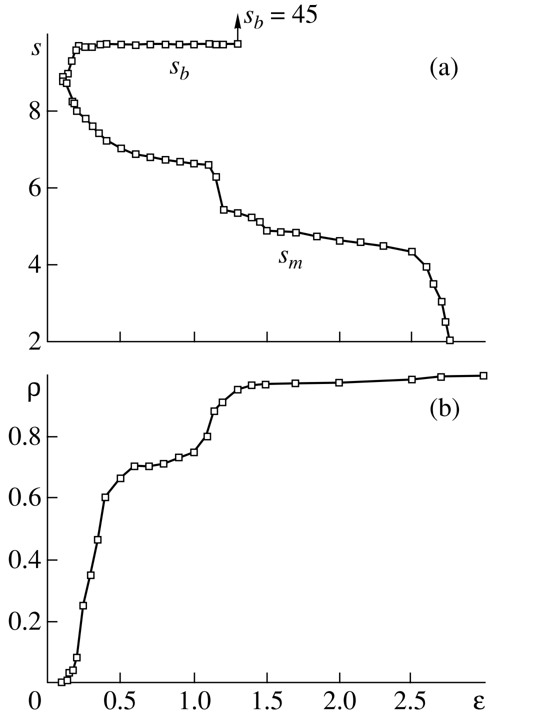

Fig. 2. Plots of (a) the lower (s_m) and upper (s_b) boundaries of the interval of synchronized time scales and (b) the relative energy ρ corresponding to the interval of synchronized time scales versus coupling parameter ε for two coupled Rössler systems.

sponding to the time scale s_2 in the master system is small, while that in the slave system is large and the former system cannot induce the corresponding phase dynamics in the latter system, so that the time scales s_2 remain nonsynchronized. As the coupling parameter grows, the interval of synchronized time scales $[s_m, s_b]$ expands and, when all time scales are synchronized, the combined system exhibits lag synchronization.

This pattern of synchronization of the time scales is qualitatively illustrated in Fig. 2a showing the plots of the upper (s_b) and lower (s_m) boundaries versus the coupling parameter ε for the interval of time scales satisfying the synchronization conditions (5) and (6). The regime of chaotic synchronization is set at $\varepsilon \simeq 0.1$, for which a certain interval $\Delta s = s_b - s_m$ appears featuring the phase entrainment. As the ε value increases, the
1 interval of synchronized timescales Δs increases and, eventually, all time scales become synchronized (lag synchronization).

A convenient characteristic for description of the degree of synchronization of two chaotic subsystems is offered by the relative energy of the wavelet spectrum corresponding to the synchronized time scales [13]:

$$\rho = \int_{s_m}^{s_b} E(s)ds \Big/ \int_0^{\infty} E(s)ds, \qquad (8)$$

where $E(s)$ is the integral distribution of the wavelet spectrum energy with respect to the time scales determined according to formula (4). The corresponding dependence is presented in Fig. 2b. As can be seen, an increase in the coupling parameter ε is accompanied by increasing fraction of the energy of the chaotic process corresponding to the synchronized time scales. For large coupling parameters ($\varepsilon > 1.5$), the relative energy $\rho(\varepsilon)$ tends to unity, which implies that almost all the energy of chaotic oscillations is concentrated in the interval of synchronized time scales and the system, as was noted above, occurs in the regime of lag synchronization. In other words, the behavior of coupled chaotic oscillators in the regime of generalized synchronization is, from the standpoint of time scale synchronization, fully analogous to their behavior in the case of phase synchronization (see [13, 14]).

Thus, various regimes of chaotic synchronization (phase, generalized, lag, and complete synchronization) are determined by synchronous behavior of a certain interval of the time scales. In other words, we may ascertain that coupled chaotic oscillators exhibit a common type of synchronous behavior called time scale synchronization, whereby all the other types of chaotic synchronization are particular cases of the time scale synchronization.

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