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Synchronization of Spectral Components of Coupled Chaotic Oscillators

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Abstract—The process of chaotic synchronization of two coupled dynamical systems with slightly different parameters has been studied. The transition from phase to lag synchronization regime is related to the fact that increasing number of spectral components in the Fourier spectra of the coupled oscillators are synchronized. For this reason, it is possible to introduce the notion of the degree of phase synchronization. A method of description of the degree of phase synchronization is proposed and it is shown that this value increases with the coupling parameter. *© 2004 MAIK "Nauka/Interperiodica".*

The phase synchronization of systems in the regime of dynamical chaos is among important problems in the modern theory of nonlinear oscillations [1, 2]. This phenomenon is described and analyzed using the concept of the phase $\phi(t)$ of a chaotic signal [1–6]. The phase synchronization implies that the phases of chaotic signals are mutually entrained, while their amplitudes remain uncorrelated and appear chaotic. The entrainment of phases leads to the coincidence of frequencies of the two signals. The frequency of a chaotic signal is defined as the average rate of phase variation,

$$\Omega = \langle \dot{\phi}(t) \rangle. \quad (1)$$

At the same time, in some cases (in particular, for the so-called systems with poorly defined phase [3, 5]), attempts at describing the phenomenon of phase synchronization in terms of the phase $\phi(t)$ may lead to incorrect results. This is related to the fact that the chaotic time series for such systems is characterized by a Fourier spectrum containing no clearly pronounced main spectral component (or there are several such components). In the case when a chaotic signal spectrum has a single pronounced frequency component, the phase $\phi(t)$ of this signal is in fact associated with this main frequency and, hence, the chaotic signal frequency must coincide with the main frequency (see [7]).

If there is no dominating frequency component in the spectrum, the phase $\phi(t)$ of a chaotic signal cannot adequately describe the system dynamics. For such systems, we have recently [8, 9 suggested to use a family of phases $\phi_s(t)$ introduced using a continuous wavelet transform [10] so that each particular phase $\phi_s(t)$ is associated with its own time scale s. In this case, the phenomenon of phase synchronization is manifested by a synchronous behavior of the phases of coupled chaotic oscillators observed on certain synchronized time scales s, for which

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \mathrm{const}. \quad (2)$$

It was demonstrated [8] that the range of synchronized scales s increases with the coupling parameter until all time sc ales will be synchronized. This corresponds to the state of lag synchronization [11], whereby the coinciding states of interacting oscillators are shifted in time relative to each other: $\mathbf{x}_1(t - \tau) \simeq \mathbf{x}_1(t)$. Further increase in the coupling parameter leads to a decrease in the time shift τ. The oscillators tend to the regime of complete (full) synchronization, $x_1(t) \simeq x_2(t)$, and the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ tends to zero on all time scales.

Thus, the family of phases introduced by a wavelet transform for a chaotic signal allows the regime of phase synchronization of coupled oscillator to be effectively revealed. On the other hand, the continuous wavelet transform is characterized by a lower frequency resolution than the Fourier transform (see [10]). The continuous wavelet transform appears as smoothening the Fourier spectrum, whereby the dynamics on a time scale s is determined not only by the spectral component $f = 1/s$ of the Fourier spectrum. This dynamics is also influenced by the neighboring components as well, the degree of this influence being dependent both on their positions in the Fourier spectrum and on the intensities. Thus, the fact that coupled chaotic oscillators exhibit synchronization on a time scale s of the wavelet spectrum by no means implies that the corresponding component $f = 1/s$ of the Fourier spectrum of these systems is also synchronized.

This study was aimed at elucidating the question as to how does synchronization of separate spectral components in the Fourier spectra of coupled chaotic oscil-

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lators proceeds depending on the coupling parameter in the phase synchronization regime.

Let $x_1(t)$ and $x_2(t)$ be the time series generated by the first and second coupled chaotic oscillators, respectively. The corresponding Fourier spectra are determined by the relations

$$S_{1,2}(f) = \int_{-\infty}^{+\infty} x_{1,2}(t)e^{-i2\pi ft}dt. \quad (3)$$

Accordingly, each spectral component f of the Fourier spectrum $S(f)$ can be brought into correspondence with an instantaneous phase $\phi_f(t) = \phi_{f0} + 2\pi ft$. However, since the phase $\phi_f(t)$ corresponding to the frequency f of the Fourier spectrum $S(f)$ linearly increases with the time, the phase difference of the interacting oscillators at this frequency, $\phi_{f1}(t) - \phi_{f2}(t) = \phi_{f01} - \phi_{f02}$ is always limited and, hence, the traditional condition of phase entrainment (used for determining the phase synchronization regime)

$$|\phi_1(t) - \phi_2(t)| < \text{const}, \quad (4)$$

is useless. Apparently, a different criterion should be used to define the phase synchronization of coupled oscillators at a given frequency f.

In the regime of lag synchronization, the behavior of coupled oscillators is synchronized on all time scales s of the wavelet transform (see [8]). Therefore, we may expect that all frequency components of the Fourier spectra of the systems under consideration should be synchronized as well. In this case, $x_1(t-\tau) \simeq x_2(t)$ and, hence, by virtue of (3), we must have $S_2(f) \simeq S_1(f)\exp(i2\pi\tau f)$. Thus, in the case of coupled chaotic oscillators occurring in the regime of lag synchronization, their instantaneous phases corresponding to the spectral component f of the Fourier spectra $S_{1,2}(f)$ will be related as $\phi_{f2}(t) \simeq \phi_{f1}(t) + 2\pi\tau f$ and, hence, the phase difference $\phi_{f2}(t) - \phi_{f1}(t)$ must obey the relation

$$\Delta\phi_f = \phi_{f1} - \phi_{f2}(t) = 2\pi\tau f. \quad (5)$$

According to this, the points corresponding to the phase difference of the spectral components of chaotic oscillators in the regime of lag synchronization on the $(f, \Delta\phi_f)$ plane must fit to a straight line sloped at $k = 2\pi\tau$ (see also [12]).[1]

It was demonstrated [8] that breakage of the lag synchronization regime (e.g., as a result of decrease in the coupling of oscillators) and transition to the regime of phase synchronization results in the loss of synchronism for a part of time scales s of the wavelet spectra. Accordingly, we may expect that a part of spectral components of the Fourier spectra in the phase synchronization regime will also lose synchronism and the points on the $(f, \Delta\phi_f)$ plane will deviate from the aforementioned straight line. It natural to assume that synchronism will be lost primarily for the spectral components accounting for a small fraction of energy, while the components corresponding to a greater energy fraction will remain synchronized and the corresponding points on the $(f, \Delta\phi_f)$ plane will be situated at the straight line. As the coupling parameter further decreases, an increasing part of the spectral components will deviate from synchronism. However, as long as the most "energetic" components remain synchronized, the coupled systems will exhibit the regime of phase synchronization.

Let us introduce a quantitative characteristic of the number of spectral components of the Fourier spectra $S_{1,2}(f)$ occurring in the regime of synchronism,

$$\sigma_L = \frac{\int_0^{+\infty} H(S_1(f) - L)H(S_2(f) - L)(\Delta\phi_{f_j} - 2\pi\tau f_j)^2 df}{\int_0^{+\infty} H(S_1(f) - L)H(S_2(f) - L) df}, \quad (6)$$

where $H(\xi)$ is the Heaviside function, L is the threshold power level (in dB) above which the spectral components are taken into account, and τ is determined by the time shift of the most energetic frequency component (f_m) in the Fourier spectra $S_{1,2}(f)$: $\tau = (\phi_{f_m2} - \phi_{f_m1})/(2\pi f_m)$. The quantity σ_L provides a measure of the degree of phase synchronization. This characteristic tends to zero in the regimes of complete and lag synchronization. In the case of phase synchronization, σ_L increases with the number of desynchronized spectral component of the Fourier spectra $S_{1,2}(f)$ of coupled oscillators.

Real data are usually represented by discrete time series of a finite length. In such cases, the continuous

[1] It is obvious that, in the regime of complete synchronization, $\mathbf{x}_1(t)f \simeq \mathbf{x}_2(t)$, the phase difference $\Delta\phi_f$ is zero for all components f of the Fourier spectrum.

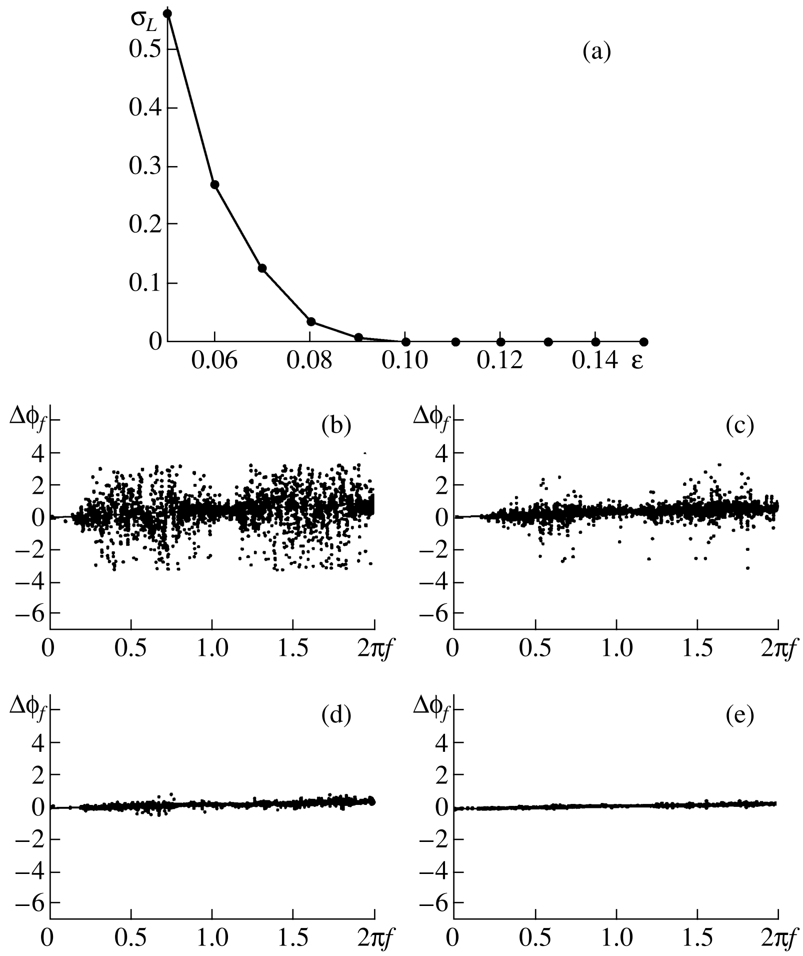

Fig. 1. (a) Plot of the degree of phase synchronization σ_L versus coupling parameter ε and (b–e) the phase difference $\Delta\phi_f$ of various spectral components f of the Fourier spectra $S_{1,2}(f)$ of two coupled Rössler systems for ε = 0.05 (b), 0.08 (c), 0.1 (d), and 0.15 (e) at a power level of L = –40 dB. The plots are constructed for the time series $x(t)$ with a length of 2000 dimensionless time units at a discretization step of h = 0.2.

Fourier transform (3) has to be replaced by its discrete analog, and the integral (6), by the sum

$$\sigma_L = \frac{1}{N}\sum_{j=1}^{N}(\Delta\phi_{f_j} - 2\pi\tau f_j)^2, \tag{7}$$

taken over all spectral components of the Fourier spectra $S_{1,2}(f)$ with the powers above L. In calculating σ_L, it is expedient to perform averaging over a set of time series [12, 13].

In order to illustrate the approach proposed above, let us consider two coupled Rössler systems in the vortex chaos regime,

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{8}$$

where ε is the coupling parameter, ω_1 = 0.98, and ω_2 = 1.03. By analogy with the case studied in [14], the values of control parameters were selected as follows: a = 0.22. p = 0.1, and c = 8.5. It is known [14] that two coupled Rössler systems with ε = 0.05 occur in the regime of phase synchronization. For ε = 0.15, the same systems exhibit lag synchronization.

Figure 1a shows a plot of the degree of phase synchronization σ_L versus coupling parameter ε. As can be seen, σ_L tends to zero when the coupling parameter increases, which is indicative of the transition from phase to lag synchronization. Figures 1b–1e illustrate the increase in the number of synchronized spectral components of the Fourier spectra $S_{1,2}(f)$ of the two coupled systems with increasing ε. Indeed, Fig. 1b corresponds to the case of weak phase synchronization (ε = 0.05), when this regime has just appeared; Figs. 1c and 1d show well pronounced phase synchronism (ε = 0.08 and 0.1, respectively); and Fig. 1e reflects the state of lag synchronization (ε = 015), whereby all spectral components are synchronized.

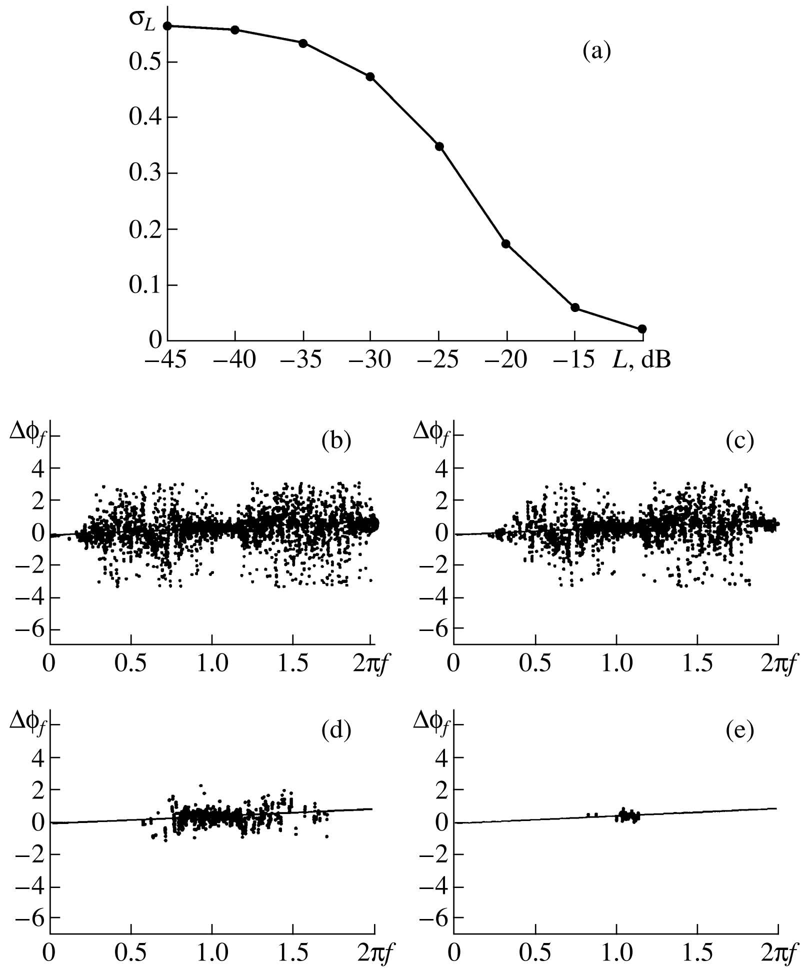

Fig. 2. (a) Plot of the degree of phase synchronization σ_L versus power L at which the spectral components f_j of the Fourier spectra $S_{1,2}(f)$ are taken into account in the formula for σ_L and (b–e) the spectra of phase difference $\Delta\phi_f$ of various spectral components f of the Fourier spectra $S_{1,2}(f)$ of two coupled Rössler systems for various power levels L = –40 dB (b), –30 dB (c), –20 dB (d), and –10 dB (e) for $\varepsilon = 0.05$.

Another important question is which spectral components of the Fourier spectra of interacting chaotic oscillators are synchronized first and which do it in the last turn. Figure 2a shows a plot of the degree of phase synchronization σ_L for $\varepsilon = 0.05$ (corresponding to weak phase synchronization) versus power L at which the spectral components f_j of the Fourier spectra $S_{1,2}(f)$ are taken into account in formula (7). As can be seen, "truncation" of the spectral components possessing small energies leads to a decrease in σ_L. Figures 1b–1e illustrate the distribution of the phase difference $\Delta\phi_f$ of the spectral components f with the power exceeding the preset level L. The data in Fig. 2 show that the most "energetic" spectral components are first synchronized upon the onset of phase synchronization. On the contrary, the components with low energies first go out of synchronism upon breakage of the lag synchronization regime.

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