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New Universality Type in Chaotic Synchronization of Dynamic Systems

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A quite universal mechanism of establishing chaotic synchronization regime in coupled dynamic systems is found. It is shown that the synchronous regime arises due to the phase coupling between the Fourier-spectrum components of the interacting chaotic oscillators. © *2004 MAIK "Nauka/Interperiodica".*

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Chaotic synchronization is one of the fundamental phenomena that have been actively studied in recent years [1] because of their theoretical and applied importance (e.g., in data transmission using the determinate chaotic oscillations [2], in biological systems [3], etc.). One can distinguish several different types of chaotic synchronization, such as generalized [4], phase [1], lag [5], and complete synchronization [6]. It was shown in [7] that the phase, lag, and complete synchronizations are closely related processes and, in essence, belong to the same type of synchronous vibrations of coupled chaotic oscillators. The character of the synchronous regime (phase, lag, or complete synchronization) is determined by the number of synchronized time scales introduced by the continuous wavelet transform [8]. Since the time scale s is associated with frequency, the synchronization of chaotic oscillations is associated with the phase coupling between the frequency components ω of the corresponding Fourier spectra $S(\omega)$.

The purpose of this work is to study the mechanism of establishing coupling between the frequency components of the coupled dynamic systems. To begin with, we consider how the close frequency components of two coupled oscillators behave when the coupling between them is strengthened. As a model of oscillators with a nearly single-frequency behavior, we choose two coupled Van-der-Pol oscillators with slightly different parameters:

$$\ddot{x}_{1,2} - (\varepsilon - x_{1,2}^2)\dot{x}_{1,2} + \Omega_{1,2}^2 x_{1,2} = \pm K(x_{2,1} - x_{1,2}), \quad (1)$$

where $\Omega_{1,2} = \Omega \pm \Delta$ are slightly different partial frequencies; $x_{1,2}$ are the variables describing the behavior of the first and second self-excited oscillators, respectively; and K is the coupling parameter. The nonlinearity parameter $\varepsilon = 0.1$ was chosen to be small to provide a nearly single-frequency character of the self-excited oscillators; the asymmetric coupling provides the establishment of synchronous regime in system (1), analogous to the lag-synchronization regime in chaotic systems, where the oscillators have the same frequency with a small phase difference that decreases with increasing coupling parameter.

Applying the method of slowly varying amplitudes, we seek the solution to Eq. (1) in the form $x_{1,2} = A_{1,2}e^{i\omega t} + A_{1,2}^{*}e^{-i\omega t}$ with $\dot{A}_{1,2}e^{i\omega t} + \dot{A}_{1,2}^{*}e^{-i\omega t} = 0$, where * stands for complex conjugation and ω is the oscillation frequency in system (1). After averaging over the rapidly varying variables, we obtain the equations for the complex amplitudes:

$$\begin{aligned}\dot{A}_{1,2} &= \frac{1}{2}(\varepsilon - |A|^2)A \\ &+ i\frac{1}{2\omega}[(\Omega_{1,2}^2 - \omega^2)A_{1,2} \mp K(A_{2,1} - A_{1,2})].\end{aligned} \quad (2)$$

By choosing the complex amplitude in the form

$$A_{1,2} = r_{1,2}e^{\varphi_{1,2}}, \quad (3)$$

we arrive at the equations for the amplitudes $r_{1,2}$ and phases $\varphi_{1,2}$ of the coupled oscillators:

$$\begin{aligned}\dot{r}_{1,2} &= \frac{1}{2}(\varepsilon - |r_{1,2}|^2)r_{1,2} \pm \frac{Kr_{2,1}}{2\omega}\sin(\varphi_{1,2} - \varphi_{2,1}), \\ \dot{\varphi}_{1,2} &= \frac{\Omega_{1,2}^2 - \omega^2 \pm K}{2\omega} \pm \frac{Kr_{2,1}}{2\omega r_{1,2}}\cos(\varphi_{1,2} - \varphi_{2,1}).\end{aligned} \quad (4)$$

The condition for the synchronous vibrations of oscillators (1) at frequency ω is that the derivatives $\dot{r}_{1,2}$ and $\dot{\varphi}_{1,2}$ be zero. Assuming that the phase difference $\Delta\varphi = \varphi_2 - \varphi_1$ is small and retaining only the terms first order in $\Delta\varphi$ in Eq. (4), one obtains the following expressions for the phase difference and frequency:

$$\Delta\varphi_{1,2} = \frac{\varepsilon\sqrt{\Omega^2 + \Delta^2 \pm 2\sqrt{\Omega\Delta(K + \Omega\Delta)}}}{2K + 4\Omega\Delta} \quad (5)$$

$$\omega_{1,2} = \sqrt{\Omega^2 + \Delta^2 \pm 2\sqrt{\Omega\Delta(K + \Omega\Delta)}}, \quad (6)$$

which correspond to the stable and unstable solutions to the system of Eqs. (4). One can see from Eqs. (5) and (6) that, at small detunings Δ, the phase difference $\Delta\varphi$ between the coupled oscillations at frequency ω is directly proportional to the oscillation frequency ω and inversely proportional to the coupling parameter K:

$$\Delta\varphi = \frac{\varepsilon\omega}{2K}. \quad (7)$$

One can see from this expression that the close Fourier components of the coupled oscillators with slightly different parameters are locked-in, with the phase difference between them given by Eq. (7). It is significant that the time lag between the spectral components

$$\tau = \frac{\Delta\varphi}{\omega} \sim K^{-1} \quad (8)$$

is independent of frequency and, hence, is the same for all spectral components. It is this fact that is responsible for the occurrence of the lag-synchronization regime, for which the dynamics of coupled chaotic oscillators have the same time lag for all frequencies. Relationship (8) is valid for many dynamic systems and, in all likelihood, has a universal character. Let us consider the manifestations of this relationship in some typical synchronization processes occurring in coupled chaotic systems.

We first consider the chaotic synchronization of two unidirectionally coupled self-excited oscillators [9, 10]. The driving oscillator is described by the system of dimensionless differential equations

$$\dot{x}_1 = -\nu_1[x_1^3 - \alpha x_1 - y_1], \qquad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1, \quad (9)$$

and the driven oscillator is described, correspondingly, by

$$\dot{x}_2 = -\nu_2[x_2^3 - \alpha x_2 - y_2] + \nu_2 K(x_1 - x_2), \qquad \dot{y}_2 = x_2 - y_2 - z_2, \quad \dot{z}_2 = \beta y_2, \quad (10)$$

where $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$ are the dynamic variables characterizing the states of the driving and driven oscillators, respectively. The controlling parameters were chosen to be $\alpha = 0.35$, $\beta = 300$, $\nu_1 = 100$, and $\nu_2 = 125$. The difference between ν_1 and ν_2 provides a slight nonidentity of the oscillators.

The time lag τ between the time realizations of coupled oscillators is shown in Fig. 1 as a function of the coupling parameter K. In this range of coupling parameters, the lag-synchronization regime prevails. One can clearly see that the time lag obeys the power law $\tau \sim K^n$ with $n = -1$, in accordance with theoretical expression (8).

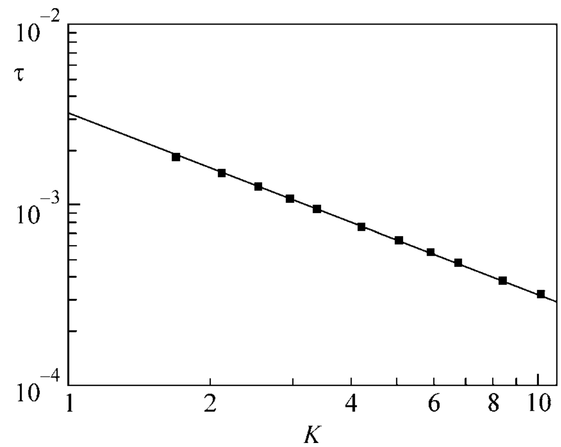

Fig. 1. (■) Time lag τ as a function of the coupling parameter K for two unidirectionally coupled chaotic oscillators (9) and (10) with slightly different parameters. The straight line on the log–log scale corresponds to the power law $\tau \sim K^{-1}$.

As a second example, we consider two coupled Ressler systems in the dynamic chaos regime:

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + K(x_{2,1} - x_{1,2}), \qquad \dot{y}_{1,2} = \omega_{1,2}x_{1,2} + ay_{1,2} + K(y_{2,1} - y_{1,2}), \qquad \dot{z}_{1,2} = p + z_{1,2}(x_{1,2} - c), \quad (11)$$

where ε is the coupling parameter, $\omega_1 = 0.98$, and $\omega_2 = 1.03$. The controlling parameters were chosen to be $a = 0.22$, $p = 0.1$, and $c = 8.5$. For the coupling parameter $0.04 \lesssim K \lesssim 0.14$, systems (11) are in the phase-synchronization regime, and at $K > 0.14$, they occur in the lag-synchronization regime (see [11]).

The time lag τ between the main Fourier components of the interacting chaotic oscillators is shown in Fig. 2 as a function of the coupling parameter K. The main frequency in the spectrum is close to $\omega = 1$ and slightly changes with an increase in the coupling parameter. After the main spectral components of the interacting oscillators are locked-in (this corresponds to the phase-synchronization regime; see also [7]), the time lag between them obeys universal power law (8).

Thus, chaotic synchronization of the coupled oscillators proceeds as follows. At a certain value of the coupling parameter, the main spectral components are locked-in, which corresponds to the phase-synchronization regime. If this main frequency component dominates the Fourier spectrum, the phase-synchronization regime can easily be tested by the traditional methods using so-called continuous chaotic-signal phase (see [1, 12]); otherwise, different methods should be used (see [7]). As the coupling parameter increases, an increasing number of spectral components become locked-in, with

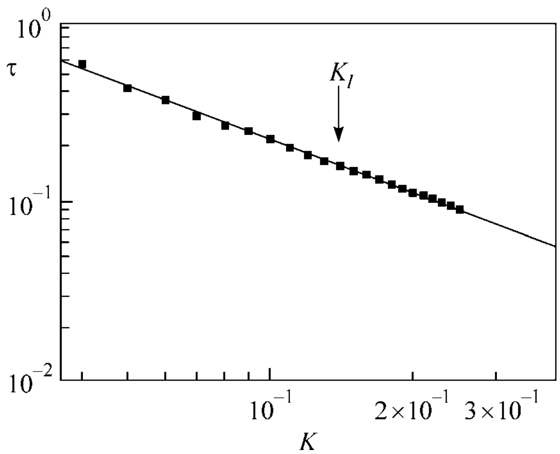

Fig. 2. (■) Time lag τ between the main Fourier components of interacting chaotic oscillators (11) vs. the coupling parameter *K*. The straight line corresponds to the power law $\tau \sim K^{-1}$. The arrow indicates the value of coupling parameter $K_l \approx 0.14$ corresponding to the onset of the lag-synchronization regime.

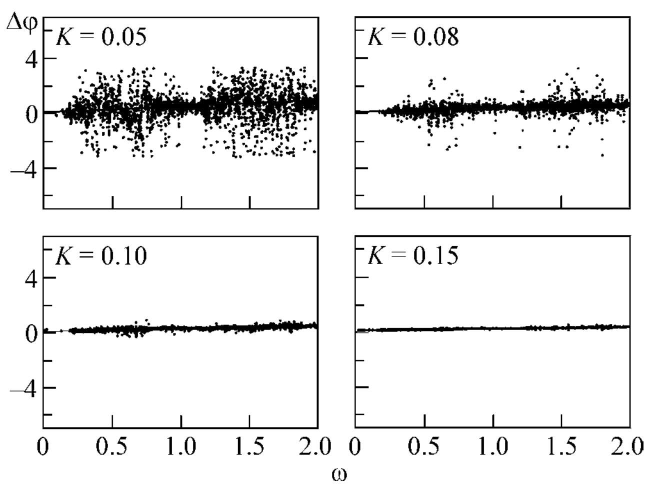

Fig. 3. Phase difference Δφ for the different Fourier components of the coupled Ressler systems. Since the phase difference between the synchronized spectral components obeys Eq. (7), the value of Δφ for the locked-in frequencies should fall on a straight line. One can clearly see that, as the coupling parameter increases, more and more spectral components become locked-in.

the time lag obeying universal dependence (8) for all of them. This is shown in Fig. 3, which demonstrates that the number of synchronized spectral components of the coupled systems increases with increasing the coupling parameter *K*. The synchronization of the individual spectral components is seen from the fact that the corresponding phase difference between these components obeys Eq. (7), while the corresponding point at the (Δφ, ω) plane falls on a straight line.

The locking-in of all frequency components corresponds to the lag-synchronization regime. As the coupling strength further increases, the time lag τ tends to zero, according to Eq. (8), and the coupled oscillations tend to the regime of complete synchronization.

Thus, we have revealed a rather universal mechanism for establishing the regime of chaotic synchronization in coupled dynamic systems. The mechanism is based on the appearance of phase coupling between the frequency components of the Fourier spectra of the interacting chaotic oscillators. The obtained results can serve as a criterion for the existence (or nonexistence) of the lag-synchronization regime in the coupled dynamic systems.

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REFERENCES

1. A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Univ. Press, Cambridge, 2001).
2. A. S. Dmitriev and A. I. Panas, *Dynamical Chaos. New Information Carriers for Communication Systems* (Fizmatlit, Moscow, 2002) [in Russian].
3. R. C. Elson *et al.*, Phys. Rev. Lett. **81**, 5692 (1998).
4. N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
5. M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
6. L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).
7. A. A. Koronovskiĭ and A. E. Hramov, Pis’ma Zh. Éksp. Teor. Fiz. **79**, 391 (2004) [JETP Lett. **79**, 316 (2004)].
8. A. A. Koronovskiĭ and A. E. Khramov, *Continuous Wavelet Analysis and Its Applications* (Fizmatlit, Moscow, 2003) [in Russian].
9. G. P. King and S. T. Gaito, Phys. Rev. A **46**, 3092 (1992).
10. A. A. Koronovskiĭ, A. E. Hramov, and I. A. Khromova, Pis’ma Zh. Tekh. Fiz. **30**, 69 (2004) [Tech. Phys. Lett. **30**, 291 (2004)].
11. M. G. Rosenblum *et al.*, Phys. Rev. Lett. **89**, 264102 (2002).
12. T. E. Vadivasova and V. S. Anishchenko, Radiotekh. Élektron. (Moscow) **49**, 76 (2004).

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