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Wavelet Transform Analysis of the Chaotic Synchronization of Dynamical Systems

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A new method for analyzing chaotic synchronization is proposed. It is based on the introduction of the family of phases for a chaotic signal using a continuous wavelet transform. The method is used to study the synchronization of two chaotic dynamical systems with ill-defined phases. © *2004 MAIK "Nauka/Interperiodica".*

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Phase synchronization [1, 2] of systems in the chaotic dynamical regime is one of the most important phenomena in the modern theory of nonlinear oscillations. Phase synchronization was experimentally observed for radio generators [3], lasers [4], electrochemical oscillators [5], heart rate [6], gas discharge [7], etc. (see also reviews [2, 8, 9]). It is quite important to study the chaotic synchronization in the information transmission by the deterministic chaotic oscillations [10].

When describing and analyzing phase synchronization, one usually introduces the phase $\phi(t)$ of a chaotic signal [1, 2, 8, 9]. Phase synchronization means that the phases of chaotic signals are locked, whereas their amplitudes remain uncorrelated and chaotic. Phase locking leads to the coincidence of the signal frequencies. The frequency of a chaotic signal is defined as the average rate of phase variation $\langle\dot{\phi}(t)\rangle$.

At present, no universal method of introduction of the phase of chaotic signal exists that would be suitable for any dynamical systems. There are several methods of phase introduction that are suited to "good" systems with a simple topology of chaotic attractor. First, the phase $\phi(t)$ of a chaotic signal is introduced as the angle in the polar coordinate system on the (x, y) plane [11]:

$$\phi(t) = \arctan\frac{y(t)}{x(t)}. \tag{1}$$

In this case, all trajectories of the projection of chaotic attractor on the (x, y) plane must rotate about the origin of coordinates.

Second, to define the phase for a chaotic dynamical system, the analytical signal [1, 8]

$$\zeta(t) = x(t) + j\tilde{x}(t) = A(t)e^{j\phi(t)} \tag{2}$$

is introduced, where the function $\tilde{x}(t)$ is the Hilbert transform of the time realization $x(t)$:

$$\tilde{x}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}d\tau. \tag{3}$$

Correspondingly, the phase $\phi(t)$ of chaotic signal $x(t)$ is determined from Eqs. (2) and (3).

Third, to define the phase of a chaotic signal, the surface of the Poincaré section [1, 2, 8] is used, and the phase is defined as

$$\phi(t) = 2\pi\frac{t-t_n}{t_{n+1}-t_n} + 2\pi n, \quad t_n \le t \le t_{n+1}, \tag{4}$$

where t_n is the time corresponding to the nth intersection of the phase trajectory and the surface of Poincaré section.

All these approaches give similar results for good systems [1, 2, 8, 9]. At the same time, they give contradictory results for systems with ill-defined phase (see, e.g., [2, 12, 13]). The traditional methods, as a rule, fail to reveal the presence of phase synchronization. For this reason, phase synchronization in such systems can be revealed by indirect measurements [12], in particular, by the calculation of Lyapunov exponents [2, 8, 14].

In this work, we consider a new method of revealing phase synchronization in dynamical systems with ill-defined phase. The behavior of such systems can be characterized by a continuous phase set defined on the basis of the following continuous wavelet transform [15] of the chaotic signal $x(t)$:

$$W(s, t_0) = \int_{-\infty}^{+\infty} x(t)\psi^*_{s,t_0}(t)dt, \tag{5}$$

where the asterisk means the complex conjugation and

$$\psi_{s,t_0}(t) = \frac{1}{\sqrt{s}}\psi_0\left(\frac{t-t_0}{s}\right) \quad (6)$$

is the wavelet function obtained from the mother wavelet $\psi_{s,t_0}(t)$. The time scale s determines the width of the wavelet $\psi_0(t)$, where t_0 is the time shift of the wavelet function. We note that the notion of the time scale is used in the wavelet analysis instead of the notion of frequency in the Fourier analysis.

As the mother wavelet, we use the Morlet wavelet [15, 16]

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}}\exp(j\omega_0\eta)\exp(-\eta^2/2). \quad (7)$$

The wavelet parameter $\omega_0 = 2\pi$ ensures the relation $s \simeq 1/f$ between the time scale s and frequency f of the Fourier transform.

The wavelet surface

$$W(s, t_0) = |W(s, t_0)|e^{j\phi_s(t_0)} \quad (8)$$

characterizes the behavior of the system for every time scale s at any time t_0. The quantity $|W(s, t_0)|$ characterizes the presence and the intensity of the corresponding time scale s at time t_0. It is also convenient to introduce the integral energy distribution over the time scales in the wavelet spectrum:

$$E(s) = \int |W(s, t_0)|^2 dt_0. \quad (9)$$

The phase $\phi_s(t) = \arg W(s, t)$ also proves to be naturally defined for every time scale s. In other words, the behavior of each time scale s can be characterized using the associated phase $\phi_s(t)$.

Let us consider the behavior of two different coupled chaotic oscillators. If these oscillators are not in the phase-synchronization regime, their behaviors are asynchronous on all time scales s. As soon as the dynamical systems under consideration are synchronized on some time scales (e.g., upon an increase in the coupling parameter between the systems), the phase-synchronization regime arises. The time scales accounting for the greatest fraction of the wavelet-spectrum energy $E(s)$ are, obviously, synchronized first. The other time scales remain unsynchronized. In this case, phase synchronization leads to phase locking on the synchronized time scales s:

$$|\phi_{s1}(t) - \phi_{s2}(t)| < \text{const}. \quad (10)$$

Here, $\phi_{s1,2}(t)$ are the continuous phases of the first and second oscillators, respectively, corresponding to the synchronized time scales s.

The approach based on the continuous wavelet transform can successfully be applied to any dynamical systems, including systems with ill-defined phase. As

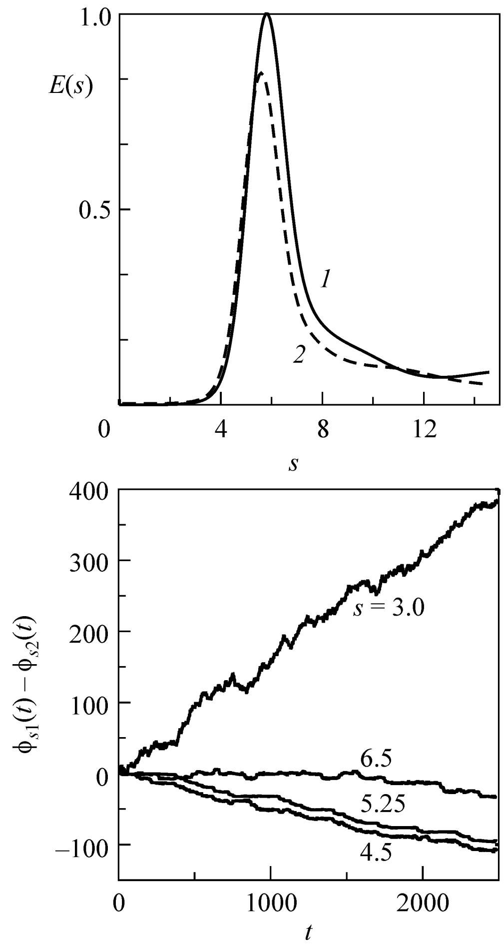

Fig. 1. Normalized energy spectrum $E(s)$ of the wavelet transform for the (*1*) first and (*2*) second Rössler systems, and the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ between two coupled Rössler systems with the coupling parameter $\varepsilon = 0.025$. The synchronization regime is absent.

an example, we consider the behavior of two different coupled Rössler systems [17] in the spiral-chaos regime:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= p + z_{1,2}(x_{1,2} - c), \end{aligned} \quad (11)$$

where ε is the coupling parameter, $\omega_1 = 0.98$, $\omega_2 = 1.03$, $a = 0.22$, $p = 0.1$, and $c = 8.5$.

Figure 1 illustrates the behavior of two coupled Rössler systems for a small coupling parameter $\varepsilon = 0.025$. The energy spectra $E(s)$ of the wavelet transform for the first and second systems are different (Fig. 1). However, the maximum energy occurs at approximately the same time scale s for both cases. According to Fig. 1, the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ increases infinitely for all time scales. This means that the sys-

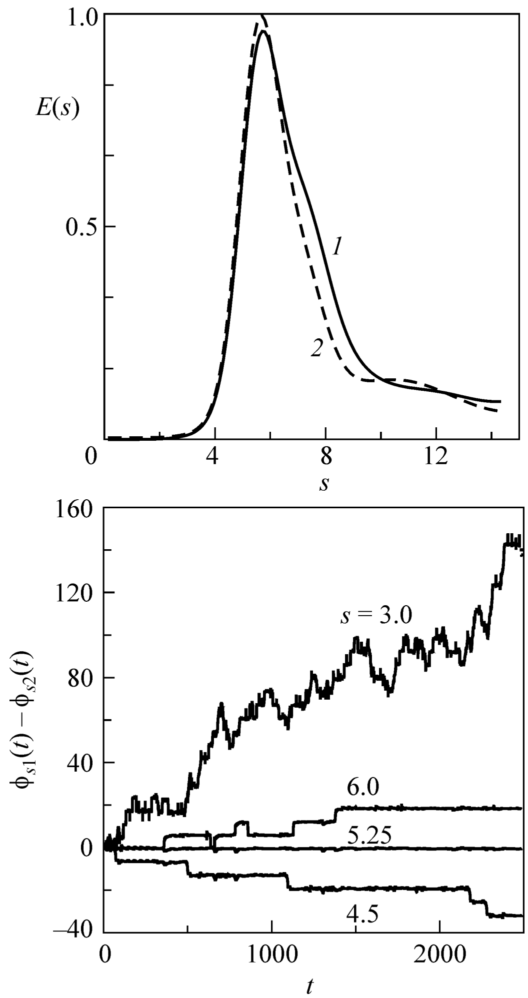

Fig. 2. The same as in Fig. 1 but for the coupling parameter ε = 0.05. Both systems are synchronized with each other on the time scales $s = 5.25$, and phase-synchronization regime prevails in the systems.

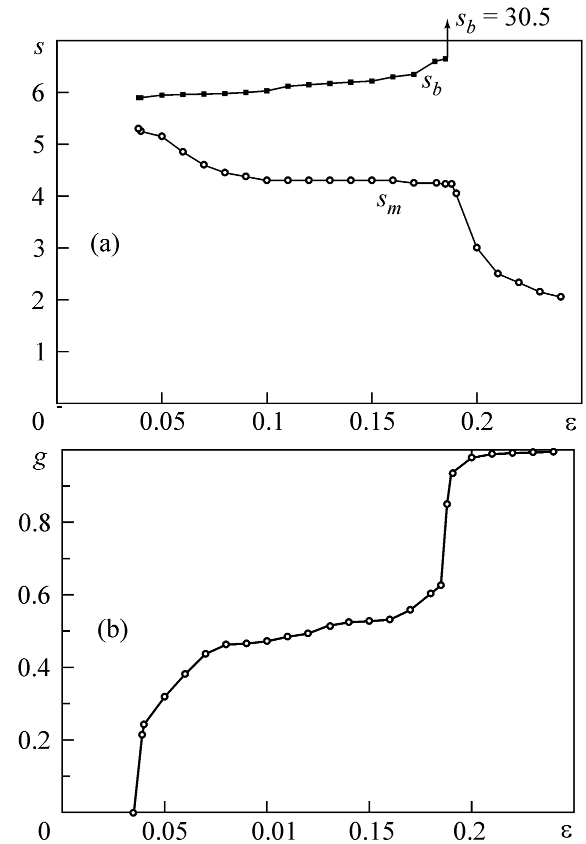

Fig. 3. (a) The upper s_b and lower s_m boundaries of the region of synchronized scales, and (b) the energy fraction γ falling on the synchronized scales for the Rössler system vs. the coupling parameter ε.

tems under consideration do not involve synchronized time scales. Therefore, the systems are unsynchronized.

As the coupling parameter increases, the systems are brought to the phase-synchronization regime (see, e.g., [11]). In particular, indirect measurements [12] show that two coupled Rössler systems with the coupling parameter ε = 0.05 are in the phase-synchronization regime. Figure 2 shows the phase difference $\phi_{s1}(t) - \phi_{s2}(t)$ for this case. It is seen that phase locking occurs at the time scale $s = 5.25$ corresponding to the maximum energy in the wavelet spectrum $E(s)$ (Fig. 2). Thus, two Rössler systems are synchronized with each other at the time scale $s = 5.25$ and, simultaneously, at close time scales. It is significant that strongly differing time scales (e.g., $s = 4.5$, 6.0, etc.) remain unsynchronized, and phase locking is not observed for these scales (see Fig. 2 and cf. Fig. 1).

With a further increase in the coupling parameter, the hitherto unsynchronized time scales become synchronized. The number of time scales for which the phase locking occurs increases. At the same time, some time scales remain unsynchronized. With a further increase in the coupling parameter ϵ, the number of synchronized time scales increases, and, when all time scales are synchronized, the lag synchronization regime prevails in the system [18].

The above qualitative picture of time-scale synchronization is illustrated in Fig. 3a, in which the upper s_b and lower s_m boundaries of the scale range where synchronization condition (10) is met are shown as functions of the coupling parameter ε. The chaotic phase synchronization arises at ε = 0.039, when phase locking occurs in a certain scale range $\Delta s = (s_b - s_m)$. Further, with an increase in ε, the range Δs of synchronized scales expands until all scales become synchronized (lag synchronization regime).

It is convenient to characterize the degree of chaotic synchronization of two chaotic subsystems by the energy fraction falling on the synchronized scales in the wavelet spectrum:

$$\gamma = \int_{s_m}^{s_b} E(s)ds / \int_0^{\infty} E(s)ds, \quad (12)$$

where $E(s)$ is the integral energy distribution over scales (9) for the wavelet spectrum. The corresponding $\gamma(\varepsilon)$ dependence is shown in Fig. 3b. As is seen, the energy fraction falling on the synchronized scales of the chaotic oscillatory process increases with an increase in the coupling parameter. For the coupling parameter $\varepsilon = 0.039$ corresponding to the appearance of the phase synchronization regime, the energy fraction falling on the synchronized scales is $\gamma = 0.21$. For large coupling parameters ($\varepsilon > 0.2$), γ tends to unity, which means that the whole energy of chaotic oscillations falls on the synchronized scales, and, as was mentioned above, the lag synchronization regime prevails in the system.

In conclusion, several important comments are noteworthy. First, the traditional approaches (1)–(4) of revealing the phase-synchronization regime through the introduction of a phase of the chaotic signal can be used for the analysis of time series characterized by the Fourier spectrum with a pronounced main frequency f_0. In this case, the phase ϕ_{s0} introduced for the time scale $s_0 \simeq 1/f_0$ approximately coincides with the phase $\phi(t)$ introduced for the chaotic signal by the traditional method given by Eqs. (1)–(4). Indeed, since the other frequencies (or other time scales) do not make a tangible contribution to the Fourier spectrum, the phase $\phi(t)$ of the chaotic signal is close to the phase $\phi_{s0}(t)$ of the main frequency component f_0 (and, correspondingly, of the main time scale s_0). In this case, the average frequencies $\bar{f} = \langle\dot{\phi}(t)\rangle$ and $\bar{f}_{s0} = \langle\dot{\phi}_{s0}(t)\rangle$ must coincide with each other and with the main Fourier frequency f_0 (see [13]):

$$\bar{f} = \bar{f}_{s0} = f_0. \quad (13)$$

If the chaotic time realization is characterized by the Fourier spectrum without a pronounced main spectral component, the traditional approaches given by Eqs. (1)–(4) do not apply. In this case, the behavior of the system must be considered at various time scales. However, such a consideration is impossible with the use of instantaneous phase $\phi(t)$ introduced for the chaotic signal by Eqs. (1)–(4). In contrast, the approach proposed above on the basis of continuous wavelet transform with the introduction of a continuous phase set can be successfully used for any type of chaotic signal.

This approach can also be used to analyze the experimental data, because it does not require any *a priori* information on the system under study. Moreover, the wavelet transform can reduce the noise effect in some cases [15, 19]. The above method can likely be useful and efficient for analyzing the time series generated by physical, biological, physiological, and other systems.

Thus, a new method for describing the chaotic synchronization is proposed. It is based on the continuous wavelet transform and analysis of the system dynamics on various time scales. This approach applies to any chaotic systems and experimental time series.

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