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PHYSICS

On the Mechanism of the Breakdown of Complete Chaotic Synchronization

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With the example of coupled Rössler systems, we consider the mechanism responsible for the breakdown of the complete chaotic synchronization of two coupled chaotic dynamical systems when varying the coupling parameter. A new method of introducing the phase of a chaotic signal is proposed on the basis of the continuous wavelet transform.

The phenomenon of synchronization of self-oscillatory systems often occurs in nature [1–3]. In recent years, researchers have placed particular emphasis on the chaotic synchronization of dynamical systems, including complete synchronization [4, 5], lag synchronization [6], global synchronization [7, 8], and phase synchronization [9, 10]. Complete synchronization implies that the states of interacting chaotic subsystems either coincide ($\mathbf{r}_1 = \mathbf{r}_2$) if these systems are identical or are close to each other ($|\mathbf{r}_1 - \mathbf{r}_2| \approx 0$) if the control parameters of the systems are slightly different. Phase synchronization means that the phase locking of chaotic signals $|m\phi_1 - n\phi_2| < \text{const}$ ($m, n \in \mathbb{R}$) is observed. The phase of a chaotic signal can be introduced in various ways [3, 10] with certain restrictions [3, 11]. The amplitudes of chaotic signals are uncorrelated in the phase synchronization mode.

In coupled slightly different chaotic oscillators, phase synchronization is observed even for weak coupling. For strong coupling, complete synchronization is realized. For intermediate coupling, lag synchronization is established when the states of two subsystems become nearly identical if one of them is shifted by a certain time lag τ: $\mathbf{r}_1(t) \approx \mathbf{r}_2(t-\tau)$. Thus, decreasing the coupling strength between two different chaotic systems, we can change complete synchronization to phase synchronization.

The purpose of this study is to reveal the mechanism responsible for the transition from one type of synchronization to another as the coupling strength between the chaotic subsystems increases (decreases). We investigate two different coupled Rössler systems:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2}, \\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c). \end{aligned} \quad (1)$$

Here, the control parameters are chosen as $a = 0.165$, $f = 0.2$, and $c = 10$ by analogy with [6]. The control parameter $\omega_{1,2} = \omega_0 \pm \Delta$ specifies the slight difference between the subsystems under consideration ($\omega_0 = 0.97$ and $\Delta = 0.02$), and ε is the coupling parameter. For $\varepsilon = \varepsilon_p \simeq 0.036$, phase synchronization is established in the coupled Rössler systems. For $\varepsilon = \varepsilon_l \simeq 0.14$, the transition to lag synchronization takes place. If the coupling parameter ε increases further, the time shift τ tends to zero, which corresponds to complete synchronization [6].

Chaotic synchronization is usually studied by calculating the Lyapunov exponents, finding the phases of chaotic signals for every subsystem and the relation between them, determining similarity functions [6], etc. Although all these methods make it possible to determine the existence of a certain type of chaotic synchronization, they do not reveal the mechanism of its initiation and the change of one type of synchronization to another. In this study, we apply the continuous wavelet transform [12] for this purpose and, on its basis, introduce the family of chaotic-signal phases.

The continuous wavelet transform is represented by the convolution

$$W(s, t_0) = \int_{-\infty}^{+\infty} f(t)\psi^*_{s,t_0}(t)dt \quad (2)$$

where the asterisk denotes complex conjugation, $f(t)$ is the function under analysis, and the two-parameter wavelet function $\psi_{s,t_0}(t)$ is derived from the mother wavelet $\psi_0(t)$as

$$\psi_{s,t_0}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-t_0}{s}\right). \quad (3)$$

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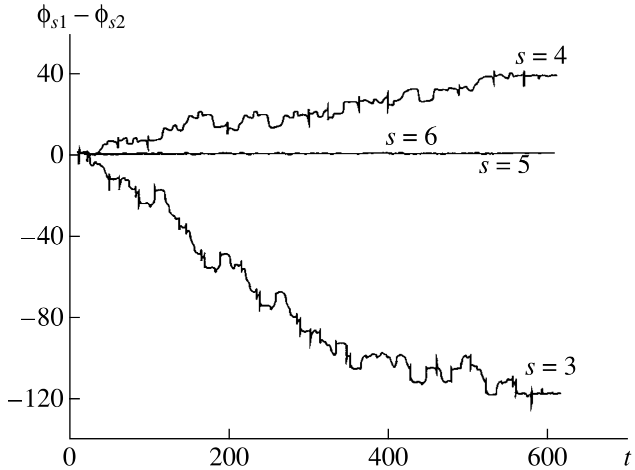

Time dependence of the phase difference $\phi_{s1} - \phi_{s2}$ for various time scales s. The time scales $s = 3$ and 4 are desynchronized, while the time scales $s = 5$ and 6, carrying a considerable part of the energy, are synchronized.

The width parameter $s \in \mathbb{R}^+$ is called the time scale of the wavelet transform, and t_0 is the shift parameter determining the position of the wavelet on the time axis t. We used the Morlet wavelet

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}}\exp(j\omega_0\eta)\exp\left(\frac{-\eta^2}{2}\right) \qquad (4)$$

with the parameter $\omega_0 = 6$. For this wavelet parameter, the time scale s corresponds to the frequency component $f \simeq \frac{1}{s}$ in the Fourier spectrum of the time realization under analysis.

The complex function $W(s, t_0)$ determined by transform (2) characterizes the dynamics of the system for the time scale s at the current time t_0. The absolute value $|W(s, t_0)|$ characterizes the contribution of the given time scale s to the time realization of the system at the current time t_0. The Morlet wavelet conditionally represents the spectral components $f \simeq \frac{1}{s}$ in the Fourier spectrum at the time t_0 and its intensity [12].

Simultaneously, the phase $\phi_s(t_0) = \arg W(s, t_0)$ is naturally specified for each time scale s. The set of such phases forms the family of chaotic-signal phases. This approach enables us, first, to determine the phase dynamics of the system for all the time scales and, second, to avoid the difficulties [3, 11] arising in conventional methods of determining the phase (based on the Hilbert transform, the Poincaré section method, etc.).

Now, we consider variations in the wavelet surfaces $W_1(s, t)$ and $W_2(s, t)$ [obtained for the time realizations $y_1(t)$ and $y_2(t)$, respectively] with a decrease in the coupling parameter ε in system (1).

For large coupling parameters, the behaviors of both systems (1) are virtually identical ($\mathbf{r}_1 \approx \mathbf{r}_2$), which is evidence of complete synchronization.[1] Correspondingly, the wavelet surfaces for complete synchronization are also identical $\{W_1(s, t) \approx W_2(s, t)\}$. It is clear that the dynamics of phases is the same for all the time scales; i.e., $\phi_{s1}(t) \approx \phi_{s2}(t)$.

When the coupling parameter ε decreases and lag synchronization arises in the system, the time realizations created by dynamical systems (1) are shifted relative to each other by a time lag τ, which increases with a decrease in the coupling parameter ε. Since $\mathbf{r}_1(t-\tau) \approx \mathbf{r}_2(t)$ and due to the definition of wavelet transform (2), the wavelet surfaces $W_1(s, t)$ and $W_2(s, t)$ are related as $W_1(s, t-\tau) \approx W_2(s, t)$. Similarly, the phases are shifted relative to each other for all time scales s; i.e., $\phi_{s1}(t - \tau) \approx \phi_{s2}(t)$. In other words, phase locking takes place for each time scale in the lag-synchronization mode; i.e., all time scales remain synchronized in the coupled systems.

With the further decrease in the coupling parameter ε, the system passes from the lag-synchronization mode to the phase-synchronization mode. In this case, the scales s, whose role in the dynamics of the system is significant, are still synchronized; in other words, phase locking remains for these scales. At the same time, certain time scales ϕ_s go out of synchronization; i.e., the phase difference for these scales increases with time.

The figure shows the time dependence of the phase difference $\phi_{s1} - \phi_{s2}$ for various time scales s. The time scales $s = 5$ and 6 carry the greatest share of the wavelet spectral density $\int |W(s, t)|^2 dt$. As is seen in the figure, the dynamics of the phases $\phi_{1,2}$ is synchronous for these scales; i.e., phase locking is observed. At the same time, the time scales carrying a small share of the wavelet spectral density (figure shows the dependences for the time scales $s = 3$ and 4) are desynchronized, and the phase difference for these scales increases with time.

With the subsequent decrease in the coupling parameter ε, phase synchronization breaks down, and the systems pass into an asynchronous mode. In this mode, all time scales s behave asynchronously, and there is no phase locking for any of them.

Thus, the phenomenon of phase synchronization can be treated as follows. A certain part of the basic time (or frequency) scales of vibrations that carry the greatest share of energy are synchronized, whereas the remaining time scales (or frequencies) have already left the synchronization mode. Therefore, the dynamics of the systems is no longer completely synchronous, although the most important time scales are still synchronized.

[1] The vectors $\mathbf{r}_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})^T$ represent the states of first and second systems (1), respectively.

The chaotic-signal phase introduced in a certain way [3, 10] for phase synchronization is properly a phase corresponding to the fundamental frequency ω_0 in the Fourier spectrum of the signal. Therefore, phase locking and, correspondingly, phase synchronization are observed until the frequencies of vibrations carrying the greatest share of energy in the spectrum are synchronous. Furthermore, as was shown in [11], inappropriate introduction of the chaotic-signal phase for which the average frequency

$$\bar{\omega} = \left\langle \frac{d\phi(t)}{dt} \right\rangle \quad (5)$$

differs from the fundamental frequency ω_0 of the Fourier spectrum leads to incorrect results. In this connection, it is efficient and informative to analyze synchronization by using the wavelet transform and introducing a family of phases ϕ_s corresponding to various time scales s for a chaotic process.

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