An Effective Wavelet Analysis of the Transition to Chaos via Intermittency

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Abstract—A new method based on the wavelet transformation is proposed, which provides for the effective determination of the duration of laminar and turbulent stages in a time pattern of the transition to dynamic chaos via intermittency. © 2001 MAIK “Nauka/Interperiodica”.

A possible classical scenario for the passage from periodic to chaotic oscillations is the passage via a transient regime or intermittency (see, e.g., [1, 2]). In this case, a periodic time pattern generated by a dynamic system is interrupted (in the course of increase in the control parameter) by irregular (or, in other words, turbulent) motions. The classical methods used for separating the regular motion stages are based on analysis of either the “current” period or the amplitude of oscillations.

Apparently, the former approach can be applied only when a laminar stage generates a signal close to strictly regular, which is by no means always the case. Actually, a laminar stage usually represents an almost periodic motion that hinders the application of this classical method and decreases accuracy of the description. The second approach can be used only in the case when the oscillation amplitude in the chaotic region differs significantly from that in the regular regime. Otherwise, we are again facing the necessity of dealing with strictly periodic oscillations in the laminar stage of motion.

For rigorously predicting the laminar and turbulent motion stages and correctly determining their durations, we propose a method based on the wavelet transformation [3, 4]

\[ W(t, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{+\infty} x(t) \psi^*(\frac{t-t'}{s}) dt' \]

(1)

where \( x(t) \) is a time pattern, \( \psi(\eta) \) is the base wavelet (asterisk denoting complex conjugate), and \( s \) is the analyzed time scale. We have used the base wavelet of the Morlet type \( \psi(\eta) = \pi^{-1/4} e^{j\omega_0 \eta} e^{-\eta^2/2} \), representing a rapidly decaying harmonic wave [5]. The \( \omega_0 \) value was taken equal to 6, which allows for an analogy with the Fourier transformation [3], which is conveniently achieved by considering a wavelet envelop \( W \) as a function of the time \( t \) and the scale corresponding to the frequency \( f_s = 1/s \).

It was found that the wavelet surface \( W(t, f_s) \) must possess significantly different structures in the regions corresponding to the laminar and turbulent stages of motion. A weakly irregular character of the signal in the laminar stage has virtually no effect on the wavelet surface structure and, hence, does not lead to errors in determining the duration of each stage. Therefore, based on the wavelet surface structure analysis using an appropriate method, we may rather simply search for different stages or intermittent regimes.

The approach to predicting the laminar and turbulent stages can be illustrated for a conventional example representing the Lorentz model system

\[ \dot{x} = \sigma(y - x), \quad \dot{y} = rx - xz - y, \quad \dot{z} = xy - bz. \]

(2)

For the set of control parameters \( \sigma = 10, b = 8/3, \) and \( r < r^* \approx 166.07 \), the time pattern generated by the Lorentz system (e.g., for the \( x \) value considered below) represents a periodic motion. Upon exceeding the threshold level \( r^* \), the regular oscillations \( x(t) \) (laminar stage) are interrupted by chaotic “outbursts,” the duration of which increases with \( r \) until the motion becomes fully chaotic (turbulent stage). An intermittency in the Lorentz system is classified as an intermittency of type I [6].

Figure 1a shows a typical time pattern \( x(t) \) and the corresponding projection of the wavelet surface \( W(t, f_s) \) for the bifurcation parameter \( r = 166.1 \). The dark regions correspond to maxima of the surface. The curve in Fig. 1a indicates the region featuring boundary effects [3]. The wavelet surface projection displays clearly manifested regions corresponding to the laminar and turbulent stages in the given time pattern.

The wavelet surface structure corresponding to the regular stage of motion has a profile \( W(t, f_s) \) with two global maxima (corresponding to two dark bands parallel to the time axis in the projection of the \( W \) surface in Fig. 1a). This profile does not change with time.
Fig. 1. Wavelet transformation analysis of the time pattern $x(t)$ numerically calculated for the model Lorentz system according to Eqs. (2) using the 4th order Runge–Kutta method (pattern length, $N = 2^{15}$ counts; time step, $\Delta t = 0.0001$): (a) projection of the wavelet surface $\tilde{W}(x, f)$ for the bifurcation parameter $r = 166.1$ and the supercritical difference $r - r^* = 0.03$; (b–d) instantaneous energy distributions for the (b) laminar ($r = 0.36$) and (c, d) turbulent stages ($r = 0.56$ and 0.72, respectively); (e) projection of the wavelet surface for $r = 167.0$ and $r - r^* = 0.93$ (black bars at the time pattern indicate the laminar stages).

$t$ within the laminar stage of motion. This shape of the wavelet surface profile, featuring two characteristic global maxima, is related primarily to some features of the Fourier spectrum of a signal corresponding to the laminar stage. This spectrum is characterized by two dominating harmonics with the frequencies $f_1 = 41.3$
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Fig. 2. Wavelet transformation analysis of the time pattern $x_n(t)$ corresponding to a logistic mapping $x_{n+1} = \varepsilon x_n(1 - x_n)$ (pattern length, $N = 256$ counts): (a) projection of the wavelet surface $W(t, f_s)$ for the mapping parameter $\varepsilon = 1 + \sqrt{8} - 0.002$ (b–d) instantaneous energy distributions for the (b) laminar ($n = 136$) and (c, d) turbulent stages ($n = 62$ and 220, respectively).

and $f_s = 131.3$, which correspond to two maxima observed in the wavelet surface.

When the system enters the turbulent stage, the shape of the $W$ surface exhibits a dramatic variation. The transition to a chaotic dynamic regime is accompanied by the “outburst” of oscillatory phenomena on various scales, the main energy falling within the scale interval corresponding to $f_s \in (f_1, f_2)$. Note that the regions of the wavelet surface corresponding to turbulent stage are strictly defined in time.

As noted above, the wavelet surface structure does not change with time in the laminar stage of motion. The instantaneous energy distribution over the time scales (frequencies) $E(f_s)|_{t = \text{const}}$ in this stage is also time-independent (Fig. 1b). This distribution exhibits a characteristic shape with two maxima, the nature of which was discussed above (the peaks are due to the two frequencies $f_1$ and $f_2$ dominating in the Fourier spectrum corresponding to the laminar stage).

The onset of a turbulent stage is manifested by splitting of the $1/f_1$ and $1/f_2$ scales. The resulting energy spectrum exhibits a single dominating scale. Subsequently, the turbulent stage features the excitation and damping of oscillations in the system. These chaotic oscillations occur on various time scales, predominantly below $1/f_1$. The energy distribution during the turbulent stage is essentially nonstationary. The only feature observed over the entire chaotic “outburst” is the presence of a dominating scale $1/f_1 \sim 0.011$ (here, the sign “~” indicates that the position of maximum energy varies and the value 0.011 represents an average over several turbulent outbursts). The shape of the energy distribution markedly changes with time. This is illustrated by Figs. 1c and 1d showing the instantaneous energy distributions over the time scales $E(f_s)$ corresponding to the time moments $r = 0.56$ and 0.72. As seen, the two curves differ in both number and position of maxima.

As the bifurcation parameter increases, the duration of laminar stages decreases and eventually (at $r > 167.0$) the laminar motion is virtually indistinguishable in the time pattern. At the same time, the wavelet analysis successfully reveals the laminar stages with a duration of 2–3 periods of oscillation on the main frequency. This is illustrated in Fig. 1e, which shows the
results of the wavelet transformation obtained for the control (bifurcation) parameter $r = 167.0$. As seen, the laminar stages in the time pattern correspond to the characteristic features in the wavelet surface projection $W(t, f)$.

This situation does not take place only in analysis of the time patterns generated by the flow systems. For illustration, Fig. 2a shows the results of the wavelet transformation of a time pattern generated by a logistic mapping $x_{n+1} = \varepsilon x_n (1 - x_n)$ with a control parameter $\varepsilon^* - \varepsilon = 0.002$. Here, $\varepsilon^* = 1 + \sqrt{8}$ is the critical value corresponding to the tangential bifurcation of the regular cycle of period 3 (whereby the stable and unstable 3-cycles merge and vanish), while lower values of the parameter correspond to a regime with the laminar stages interrupted by turbulent outbursts.

An analysis of the wavelet surface and its comparison with the time pattern shows that laminar stages in this system also correspond to a special characteristic structure of the $W$ function. Similar to the dynamics observed for the Lorentz system, the escape from the laminar stage of motion is accompanied by the disappearance of the time scale corresponding to regular oscillations. Also significantly different are the characteristic oscillation energy distributions over the time scales (frequencies) for the chaotic and laminar regimes. In the latter case, the $E(f)$ distribution remains unchanged and has a shape depicted in Fig. 2b. In the turbulent regime, the energy distribution is time-dependent (see Figs. 2c and 2d showing the $E(f)$ functions for two different time instants).

The results of the wavelet analysis presented above for the Lorentz model exhibit a sufficiently universal character from the standpoint of analysis of the transition to chaos via intermittency. Therefore, the wavelet transformation of time patterns generated by dynamic systems featuring a transition to chaos via intermittency can be used for distinguishing and analyzing the laminar and turbulent stages.

The structure of the wavelet surface can be analyzed by a method based on determining the number of maxima $F$ in the instantaneous energy distributions over the time scales $E(f)|_{t = \text{const}}$. The number of maxima, which is unchanged during a laminar stage, becomes time-dependent upon transition into a turbulent regime. Using this approach, we may effectively separate the laminar stages of motion. Taking into account that $F$ may acquire only integer values, the determination of the duration of, for example, the laminar stage from the $F(t)$ function is very simple. This method is well applicable upon going to a greater bifurcation parameter.

To illustrate the proposed method, we have determined the average durations $l$ of laminar and turbulent stages as functions of the bifurcation parameter $r$. The number of laminar (and, accordingly, the turbulent) stages for which the $l$ values were determined amounted to 600–800. The results of this analysis are presented in Fig. 3a. The average duration of the laminar stage of motion noticeably decreases with increasing $r$; the average duration of the chaotic stage varies at a much lower rate, showing a tendency to increase with the bifurcation parameter. The latter fact is quite explainable, since the chaotic stage of the motion is essentially the stage of “relaminarization,” that is, of the system return to the regular motion. Evidently, the duration of the relaminarization process must be independent of the bifurcation parameter $r$.

Figure 3b shows the plot of $\ln l$ versus $\ln (r - r^*)$. The logarithmic scale clearly demonstrates that the laminar stage duration is proportional to the square root of the distance $r - r^*$ from the intermittency threshold (at least for sufficiently small $r - r^*$ values), which is in agreement with the theory of type I intermittency [6]. For
large values of the supercritical difference, the laminar stage duration deviates from the power law. Thus, the proposed method gives results in good agreement with the published data [2, 6].

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REFERENCES


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